Problème de Cauchy caractéristique et scattering conforme en relativité générale

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THESE DE L'UNIVERSITÉ DE BRETAGNE OCCIDENTALE

sous le sceau de l'Université européenne de Bretagne

pour obtenir le titre de

DOCTEUR DE L'UNIVERSITÉ DE BRETAGNE OCCIDENTALE

Mention: Mathématiques

École Doctorale SICMA

présentée par

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préparée au Laboratoire de Mathématiques de Brest

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Thèse soutenue le 2 juin 2010

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Introduction

Le problème de Cauchy caractéristique en relativité générale, et plus particulièrement lorsqu'il est posé sur un cône de lumière, est un problème central en relativité. Tout d'abord d'un point de vue heuristique, nous observons de l'univers un cône passé. La question se pose donc de savoir si il est possible d'extrapoler à partir de ces données ce qui se passe à l'intérieur de ce cône. Par ailleurs, un cône de lumière est une structure naturelle sur une variété lorentzienne. Le problème de Cauchy usuel suppose au contraire de considérer un problème d'évolution : ceci implique un double choix arbitraire non intrinsèque d'une surface de Cauchy pour les données initiales et d'un feuilletage temporel. D'un point de vue technique, le problème de Goursat est un outil essentiel qui intervient fréquemment en relativité :

– dans l'étude des équations d'Einstein et de leur stabilité, le problème de Cauchy caractéristique est un élément crucial ; on peut ainsi citer les travaux de Klainerman-Nicolò ([58, 57]) ou plus récemment, Nicolò ([74] ou Caciotta-Nicolò ([10]) sur le problème caractéristique pour les équations d'Einstein.
– Penrose s'interessa au début des années 1960 [77, 78] aux propriétés de radiation des solutions d'équations des ondes. Son approche par compactification conforme lui permet de décrire le comportement asymptotique des champs le long de rayons lumineux comme des propriétés de traces sur une hypersurface caractéristique à l'infini : la résolution du problème de Cauchy traduit alors le fait que que le comportement asymptotique détermine les solutions. L'approche de Penrose a été reprise dans un cadre numérique ([36]) et a donné lieu aux premières théories de scattering conformes dans le cas plat par Friedlander ([40]) puis par Baez-Segal-Zhou ([6]).

Le travail présenté ci-après est dans la lignée des travaux de Jean-Philippe Nicolas effectués sur le problème de Cauchy caractéristique pour les équations de Dirac ([49, 50, 69, 70, 71]) et sur le problème du scattering en relativité, en particulier celui de scattering conforme ([62, 63]). L'étude se concentre sur deux points :

– l'établissement d'une formule intégrale pour le problème de Cauchy caractéristique pour l'équation de Dirac généralisant au cas courbe les travaux de Penrose dans [78] ;
– un résultat de scattering conforme pour une équation des ondes non linéaire conformément invariante sur un espace asymptotiquement simple.

Nous présentons dans la suite rapidement les idées sous tendant ce travail. Dans un premier temps, nous introduisons la structure de cône de lumière, en insistant sur le formalisme qui a été utilisé pour le décrire. Il s'ensuit une brève présentation du problème de Cauchy caractéristique où sont en particulier présentées deux méthodes de résolution. Cette introduction s'achève sur un résumé des travaux effectués.

Geométrie lorentzienne et relativité

Nous présentons dans cette section le cadre et les outils géométriques des travaux qui suivent, avec un accent particulier sur la notion de structure nulle en relativité.

Dans toute cette section, on considère une variété lisse M de dimension 4 munie d'une forme bilinéaire symétrique g lisse de signature $(+ - --)$. Un tel espace est dénommé « espace-temps ». Par abus, la forme bilinéaire g est aussi appelée métrique lorentzienne (ou métrique) sur M.

M est par ailleurs munie d'une orientation temporelle.

Cadre géométrique

La structure de cône de lumière est essentielle en relativité générale à plus d'un titre. Pour le présent travail, elle joue un rôle fondamental du fait qu'elle régit la propagation des ondes et qu'elle détermine la structure conforme de l'espace-temps. On s'attache dans cette section à décrire brièvement la structure de cône de lumière sur une variété, les difficultés d'ordre géométrique et les méthodes utilisées pour les étudier.

Étant donnée la signature de la métrique, il existe dans l'espace tangent en chaque point de l'espace-temps un cône de lumière. Ce dernier est projeté sur la variété à l'aide de l'application exponentielle. On obtient alors sur M, définie localement, une notion de cône de lumière, qui, épointé en son sommet, est alors constitué de deux composantes connexes, correspondant respectivement aux images des cônes futur et passé.

Il est important de noter que cette définition du cône de lumière est une définition locale. Les limites d'existence du cône de lumière sont celles qui définissent le cut-locus au point p : l'existence de champs de Jacobi entre p et un point du bord, nuls en ces deux extrémités (la différentielle de l'exponentielle n'est pas injective) ou cisaillement de géodésiques nulles issues de p (défaut d'injectivité de l'exponentielle ; voir [12]). Cette limitation a une certaine importance dans la propagation des ondes dans un espace-temps.

L'un des outils fondamentaux de description d'une structure nulle est le formalisme de Newman-Penrose et sa version compactée, le formalisme de Geroch-Held-Penrose :

1. la première est le formalisme de Newman-Penrose sous sa forme vectorielle : il repose sur le choix d'une tétrade, dite de Newman-Penrose, formée de quatre vecteurs complexes $(l, n, m, \overline{m})$ de type lumière formant une base de $T_pM \times \mathbb{C}$, normalisée ou non ;
2. à cette tétrade correspond une unique (modulo signe) base locale (o^A, ι^A) du fibré $\mathbb{S}^A$ des spineurs à deux composantes (ou spineurs de Weyl) ; on obtient alors le formalisme spinoriel de Newman-Penrose ;
3. lorsqu'on tient compte des propriétés de transformation des objets du formalisme de Newman-Penrose par changement d'échelle des spineurs de la dyade, on obtient le formalisme de Geroch-Held-Penrose qui ne repose que sur le choix de deux directions isotropes, l et n, les deux autres étant laissées variables ; c'est un formalisme plus souple et particulièrement adapté à l'étude dynamique des congruences de géodésiques isotropes.

L'étude de la géométrie des cônes de lumière est un sujet important en relativité générale, tant comme objet fondamental (pour la propagation des ondes, pour la structure nulle à l'infini) que comme outil de description de la géométrie locale d'une variété lorentzienne (pour les feuilletages par des hypersurfaces caractéristiques, par exemple). De nombreux travaux lui ont donc été consacrés. L'approche fondatrice de Penrose, Newman-Penrose et plus particulièrement Geroch-Held-Penrose a été utilisée tout d'abord par Penrose dans son article [78] dont nous étendons les résultats dans cette thèse : il obtient en espace-temps plat une formule intégrale pour les solutions du problème de Cauchy caractéristique pour les équations d'ondes de spin arbitraire ; la dynamique des géodésiques au sommet du cône joue d'une part un rôle essentiel dans sa construction et le formalisme GHP y est d'autre part utilisé pour vérifier a posteriori que sa formule définit bien une solution. Ce formalisme est également à la base des travaux de Ehlers *et al.* ([34, 80, 35]) s'intéressant aux caustiques sur les cônes de lumière et a été développé plus avant par Frittelli-Newman *et al.* (voir par exemple [41, 42]) dans leur étude des singularités des surfaces isotropes. Des idées analogues sont également développées dans les travaux récents de Klainerman-Nicolò ([30, 59]) et Klainerman-Rodnianski ([60]).

Compactification conforme

Un problème délicat en relativité est de définir un comportement asymptotique sans recourir au choix d'une fonction de temps. Cette méthode a été introduite par Penrose dans [77] pour étudier le comportement asymptotique de l'équation des ondes sur l'espace-temps plat (ou plus précisément le peeling des solutions de l'équation des ondes, c'est-à-dire la construction d'un développement asymptotique le long de géodésiques nulles). Le principe consiste à faire un rééchelonnement de la métrique g à l'aide d'un facteur conforme Ω en introduisant la métrique $\hat{g} = \Omega^2 g$. Lorsque Ω est bien choisi, la variété $(M, \hat{g})$ se prolonge en une variété lorentzienne à bord $(\hat{M}, \hat{g})$ dont le bord est donné par $\Omega = 0$. Ce dernier est composé de deux composantes connexes $\mathcal{I}^+$ et $\mathcal{I}^-$ correspondant, respectivement, aux extrémités future et passée des géodésiques de type lumière pour la métrique g. Si l'on ajoute l'hypothèse que la différentielle du facteur conforme ne s'annule pas sur le bord, l'espace M est alors dit asymptotiquement simple. Il est important de noter que, bien que l'on parle de compactification conforme, le prolongement $\hat{M}$ n'est pas nécessairement compact.

Dans le cas particulier de l'espace de Minkowski, pour un choix pertinent de choix de facteur conforme, ce prolongement est complété par trois points i^+ et i^- représentant respectivement les extrémités sur $\hat{M}$ des courbes temporelles de longueur infinie et i^0 représentant l'infini spatial. L'espace obtenu est alors compact et a la propriété de se plonger isométriquement dans le cylindre $\mathbb{R} \times \mathbb{S}^3$ muni de la métrique lorentzienne canonique.

Un tel prolongement n'existe a priori pas pour une variété lorentzienne générique. Néanmoins, Corvino ([19]), Chrusciel-Delay ([16, 17]) et Corvino-Schoen ([20]) ont établi l'existence d'espace-temps asymptotiquement simple satisfaisant les équations d'Einstein. Leur prolongement a la particularité d'avoir une structure analogue à celle de l'espace de Minkowski. La métrique aux singularités en i^+ et i^- est la restriction d'une métrique de classe C^k (pour k entier arbitrairement grand) sur une extension de $\hat{M}$ dans un voisinage de i^+ et i^-.

Ce principe de compactification conforme de l'espace de Minkowski a été utilisé de diverses manières :

– dans [77], pour étudier le comportement asymptotique de l'équation des ondes ;
– dans [13], par Choquet-Bruhat, pour établir un résultat d'existence globale pour une équation des ondes non linéaires du type $\square u = |u|^p$;
– Baez, Segal et Zhou furent les premiers dans [6] à mettre en pratique le principe du scattering conforme issu des idées de Penrose et formalisé par Friedlander dans

[40, 38] dans l'espace-temps de Minskowski : l'opérateur de scattering est obtenu en prenant les traces des solutions de l'équation des ondes conforme ;
– prolongeant les travaux de Baez-Segal-Zhou, Mason et Nicolas prolongèrent dans [62, 63] les constructions de scattering conforme dans le cadre des espaces-temps asymptotiquement simples de Corvino-Schoen et Chrusciel-Delay.

Problème de Cauchy caractéristique

Comme il a été précisé plus haut, le problème de Goursat, ou problème de Cauchy caractéristique, a une grande importance en relativité, tant heuristique que technique. Cette section est destinée à présenter les différences de ce dernier par rapport au problème de Cauchy, ainsi que des méthodes géométriques de résolution.

Présentation du problème

Le problème de Goursat est un type particulier de problème de Cauchy pour lequel les valeurs initiales sont posées sur des données caractéristiques de l'équation aux dérivées partielles considérée. En termes d'analyse micro-locale, une caractéristique est le lieu dans le fibré cotangent des zéros du symbole principal de l'opérateur. Dans le cadre de la relativité, où l'on s'intéresse à des équations d'onde, c'est-à-dire des équations dérivées de la métrique, ce lieu d'annulation est projeté sur la variété et est alors constitué de surfaces dites caractéristiques : la restriction de la métrique à ces surface est dégénérée. C'est le cas des cônes de lumières dont un exemple important est donné par les infinis isotropes passé et futur, $\mathscr{I}^+$ et $\mathscr{I}^-$. Ils sont, en l'absence de symétrie, les surfaces caractéristiques les plus naturelles à considérer, tout en étant particulièrement délicates du fait de leurs singularités (au sommet du cône, mais aussi à l'infini spatial pour $\mathscr{I}$). Enfin, le fait de travailler avec une telle surface implique des restrictions que nous allons détaillées ci-après.

Tout d'abord, du fait de la structure géométrique des équations d'onde, l'une des principales différences du problème caractéristique avec le problème de Cauchy usuel est que le champ complet sur la surface caractéristique peut être recouvré à partir de données incomplètes, appelées données caractéristiques, et de la restriction de l'équation considérée à la surface caractéristique. De manière plus précise, ceci signifie pour les deux équations qui sont étudiées dans cette thèse :
– pour l'équation de Dirac-Weyl (pour les spineurs à Deux composantes), que l'une des composantes peut-être recouvrée à l'aide de l'autre via une équation de transport le long des générateurs de la surface caractéristique ;
– pour l'équation des ondes, qu'une dérivée transverse du champ peut-être recouvrée à partir du champ via, de la même manière que pour l'équation de Dirac, une équation de transport.

Dans les deux cas, les équations de transport sont délicates car singulières au sommet du cône. De ce fait, la résolution du problème de Goursat nécessite moins de données que le problème de Cauchy. Cette propriété reste vraie dans des cadres peu réguliers où les équations de transport ne peuvent pas être écrites (voir [53]).

Le problème de Goursat peut être mal posé du fait de la géométrie de la surface : c'est par exemple le cas lorsqu'on travaille avec un plan caractéristique dans l'espace de Minkowski. Même dans le cas où le problème peut être résolu, la résolution se fera unilatéralement dans le futur ou le passé de la surface (selon la géométrie), sauf si on travaille dans un espace-temps spatialement compact. Par un cône futur (resp. passé), l'unicité sera assurée dans le futur (resp. passé) du cône par exemple par le biais d'estimations d'énergie (voir [39] section 5.4 pour l'équation des ondes et [71, 49] pour l'équation de Dirac). Plus précisément, l'ensemble des points pour lequel le problème de Cauchy est bien posé est le domaine de dépendance de la surface caractéristique portant les données initiales.

Le présent travail étudie le problème de Cauchy caractéristique dans le cas où les données sont spécifiées sur un cône de lumière.

Deux méthodes de résolution

Il n'existe à la connaissance de l'auteur que peu de résultats généraux pour la résolution du problème de Cauchy caractéristique sur un cône :

– il existe tout d'abord l'ensemble : des techniques liées à l'analyse microlocale. Ces dernières sont issues des travaux de Leray et d'Hadamard sur le problème de Cauchy. Elles furent utilisées pour des résolutions dans des cadres très réguliers : la solution du problème est donnée par une série dont la convergence est assurée par l'analyticité des données (voir par exemple [43]). Cette méthode a été utilisée par Friedlander pour la construction d'une paramétrix de l'équation des ondes qui requiert la résolution d'une série de problèmes de Cauchy caractéristiques. Ces techniques ont ensuite été améliorées par Hörmander et Duistermaat via l'utilisation des opérateurs intégraux de Fourier ([52, 32]).
– les techniques de Hörmander présentées dans [53] qui reposent sur des estimations d'énergie.

Dans la mesure où les techniques d'analyse microlocale supposent de travailler dans un cadre très régulier, nous avons choisi de nous focaliser d'une part sur l'approche de Friedlander, récemment utilisée par Klainerman-Rodnianski dans ([60]) pour obtenir une formule de représentation de la courbure et d'autre part sur les techniques d'estimations d'énergie utilisées par Nicolas dans [73, 72] qui généralisent les travaux de [53] à des métriques Lipschitz.

Il est néanmoins important de signaler qu'il existe une littérature extensive pour des problèmes caractéristiques posés sur des surfaces de type lumière sécantes : c'est le cas des travaux de Caciutta-Nicolò ([10]) et de leurs prédécesseurs Muller zum Hagen *et al* ([15, 68, 67]) pour le problème caractéristique pour des équations d'onde tensorielles quasi linéaires et de Dossa *et al.* ([29, 54, 30]). Signalons enfin un travail récent de Dossa-Touadera ([31]) sur un problème caractéristique sur un cône de lumière.

La méthode de Friedlander, développée dans [39] pour l'équation des ondes scalaire et tensorielle sur une variété lorentzienne quelconque, est fondée sur une méthode d'approximation de la solution par une série en puissances du carré de la distance géodésique. Elle souffre de la restriction inhérente à la résolution de problème de Goursat sur un cône : ce dernier n'ayant sur la variété qu'une existence locale, la résolution ne peut s'effectuer que dans un voisinage du sommet. Ceci justifie l'usage de la distance géodésique comme fonction caractéristique du cône. Les coefficients de la série sont des fonctions satisfaisant une récurrence initialisée à l'aide de la donnée de Goursat et constituée d'une série d'équations de transport singulières au sommet du cône. La convergence de la série n'étant assurée que dans le cas analytique ([43]), cette dernière est tronquée et un terme correctif lisse est ajouté.

La méthode d'Hörmander repose sur des estimations d'énergie similaires à celles faites pour le problème de Cauchy. La particularité de ce travail repose sur la technique employée pour décrire la surface caractéristique (ou plus exactement faiblement caractéristique puis-qu'il lui est autorisé d'être de type espace ou nulle) : elle est représentée à l'aide du graphe d'une fonction. Ceci présente l'avantage de pouvoir décrire aisément les quantités caractéristiques sans avoir recours à des choix arbitraires de base dans l'espace tangent aux cônes. Mason-Nicolas, travaillant avec des feuilletages par des hypersurfaces spatiales dégénérant à l'approche de l'infini conforme, l'utilisèrent dans [62, 49] afin de mesurer la vitesse de dégénérescence du feuilletage.

Présentation du travail réalisé

Deux aspects du problème de Cauchy caractéristique sont abordés :

- le premier chapitre généralise la formule intégrale obtenue par Penrose ([78]) dans le cas plat à des espaces-temps courbes quelconques. Comme le résultat de Penrose, la formule est valable pour toutes les équations spinorielles linéaires, pourvu que ces dernières soient consistantes sur les espaces-temps considérés. La construction repose sur des formulations faibles en terme de distributions et une description précise des générateurs isotropes du cône à l'aide du formalisme GHP.
- Le second chapitre est consacré au scattering conforme pour une équation des ondes non linéaire conformément invariante. Les techniques de champs de vecteurs sont utilisées pour obtenir des estimations d'énergie qui permettent d'établir un théorème d'existence globale pour de petites données au problème de Cauchy caractéristique sur l'infini isotrope et la construction d'un opérateur de scattering conforme Lipschitz à partir des opérateurs de trace sur l'infini isotrope.

Ces deux parties ont en commun le fait que les démonstrations reposent sur une description adaptée à la géométrie du cône caractéristique.

Formule intégrale pour le problème de Cauchy caractéristique pour l'équation de Dirac

Penrose s'intéressa en 1963 ([78]) au problème de Cauchy caractéristique pour l'équation de Dirac pour obtenir des résultats de peeling pour l'équation de Dirac de spin arbitraire en espace temps plat. Il se fondait sur le formalisme spin pour décrire le comportement de congruences de géodésiques nulles, en dérivant les équations obtenus par Sachs en 1961 sur la convergence et le cisaillement d'une telle congruence nulle. Ayant posé la formule a priori pour le problème de Cauchy caractéristique, cette dernière est vérifiée en utilisant les propriétés d'invariance d'échelle associée aux composantes du spineur (c'est ainsi qu'il reformule sa démonstration dans son ouvrage avec Rindler [79], tome 1 section 5.11).

Il n'existe pas à ce jour, à la connaissance de l'auteur, de généralisation de cette formule intégrale en espace-temps courbe général. On peut cependant citer les travaux de Ottesen ([76]) avec des méthodes de type Fourier intégral, ou de Kerrick ([56]) qui retrouve la formule de Penrose à partir d'une solution fondamentale de l'équation des ondes. Ces deux auteurs travaillent en espace temps plat.

Le principe de la dérivation de la formule employé ici est fondé sur le travail de Friedlander dans son ouvrage sur l'équation des ondes ([39]) : ce dernier donne une expression d'une équation des ondes tensorielle construite à partir d'une équation des ondes scalaire. Cette construction a été adaptée au cas spinoriel en tirant partie de la structure symplectique d'une part et de la structure de module de Clifford d'autre part. L'adaptation s'est faite en plusieurs parties. Friedlander construit sa solution fondamentale pour l'équation des ondes tensorielle dans le cadre d'espace de distributions. Ce cadre a été élargi pour prendre en compte la structure de fibré de modules de Clifford, c'est-à-dire d'une part la structure de fibré vectoriel et, d'autre part, la structure de la multiplication de Clifford qui permet de définir l'opérateur de Dirac. Afin de respecter cette contrainte de structure, la construction est dans un premier temps réalisée sur les spineurs de Dirac (ou 4 spineurs) de la manière suivante :

1. s'inspirant des travaux de Günther ([46]) pour l'équation des ondes sur un fibré vectoriel et de Unterberger ([83]) qui développe une théorie des opérateurs de Fourier intégraux adaptée à la notion de spineurs, la notion de distribution à valeurs spinorielles a été développée en utilisant le produit symplectique sur le fibré des spineurs de Dirac (ou 4-spineurs) qui induit une dualité pour les spineurs lisses sur la variété.
2. D'autre part, en utilisant les travaux d'Unterberger mentionnés ci-dessus et les travaux de Dimock ([24]), on remarque que la structure symplectique possède des propriétés d'invariance conforme sous l'action de la multiplication de Clifford. Par conséquent, le produit de dualité induit par le produit symplectique permet de transporter la notion de multiplication de Clifford par un vecteur sur les distributions à valeurs spinorielles par antisymétrie. Un opérateur de Dirac est alors défini par dualité sur les distributions à valeurs spinorielles.
3. La structure de module de Clifford et d'opérateur de Dirac a alors été transporté sur les distributions à valeurs spinorielles. Il est dès lors possible de définir une notion de solution fondamentale pour l'opérateur de Dirac. Par dérivation de la solution fondamentale définie pour l'équation des ondes et en utilisant la formule de Böchner-Lichnerowicz-Schrödinger, on obtient une solution fondamentale pour l'équation de Dirac.

Il est important de noter que, suivant la méthode de Friedlander, cette construction n'est pas globale : étant fondée sur la construction de la fonction distance et sur les congruences géodésiques nulles, cette méthode est nécessairement limitée à un voisinage géodésiquement convexe du point autour duquel la résolution du problème de Goursat est effectuée.

La seconde étape est l'écriture de la formule integrale issue de l'inversion du problème de Cauchy à partir de la solution fondamentale. Cette étape repose sur deux aspects :

– le premier est l'usage des propriétés de symétrie de la multiplication de Clifford par rapport au produit de dualité ;
– le second est une étude du comportement de l'intersection de deux cônes, l'un futur, l'autre passé, lorsque le premier est fixe et que le second bouge, rendue nécessaire du fait que la solution fondamentale pour l'équation de Dirac est à support sur un cône caractéristique. La solution portée à ce problème est fondée sur les méthodes de description du cône de lumière introduites par Penrose et présentées plus haut (voir chapitre 4.14 dans [79]). Il est en particulier possible de décrire le comportement de la forme volume de l'intersection des deux cônes en fonction des coefficients de cisaillement et de convergence de la congruence géodésique nulle engendrant le cône de lumière portant les données caractéristiques.

La construction précédente est limitée aux spineurs de Dirac. Pour obtenir la formule pour un spin arbitraire, il est nécessaire d'adapter la structure de l'espace de distributions de sorte que l'espace sur lequel sera construit les distributions soit un espace symplectique muni d'une multiplication de Clifford. Le principe consiste à compléter l'espace des spineurs de rang n $\mathbb{S}_{A\ldots F}$ de sorte que l'action de la multiplication de Clifford sur la première composante laisse stable cet espace. Toute la théorie précédemment développée peut alors s'appliquer de la même manière et l'on peut obtenir une généralisation de la formule intégrale en spin arbitraire.

Il est bien sûr important de noter que cette formule en spin arbitraire se limite au cas où la géométrie de l'espace-temps est compatible avec une équation de Dirac sur un fibré de spineurs symétriques, c'est à dire lorsque la courbure satisfait certaines propriétés pour les équations de spin 1 et $\frac{3}{2}$ ou dans le cas où la variété lorentzienne est conformément plate. Dans le cas plat, le formule de Penrose est retrouvée pour tous les spins.

Scattering pour une équation des ondes non linéaires

Le second chapitre de la thèse est consacré à l'étude d'un problème non linéaire pour une équation des ondes.

Une question importante liée aux équations d'Einstein est la construction de solutions admettant une compactification conforme ; considérée en tant que problème d'évolution, cette question revient à comprendre les propriétés de scattering des solutions des équations d'Einstein. L'étude du scattering des ondes sur des espaces-temps d'Einstein est donc une première étape nécessaire. Il s'agit d'un programme qui a été initié par Dimock ([25]) et Dimock-Kay ([28, 26, 27]), puis poursuivi et développé par Bachelot ([1, 4, 2, 3]), Bachelot et Bachelot-Motet ([5]), Häfner ([47, 48]), Häfner-Nicolas ([50]), Melnyk ([64]) et Daudé [22, 23].

Toutes ces techniques reposent essentiellement sur des méthodes spectrales, ce qui requiert l'indépendance de la métrique par rapport à un paramètre de temps (c'est-à-dire une métrique statique). Lorsque la géométrie dépend du temps, d'autres méthodes doivent être développées. S'inspirant des idées de Penrose, Friedlander introduit dans[40] la notion de champ de radiation, défini par compactification conforme et correspondant au profil de scattering. Il pose là en espace-temps plat les premières bases des théories de scattering conforme que Baez-Segal-Zhoureprirent pour développer une théorie de scattering complète en espace-temps plat pour une équation des ondes non linéaire conformément invariante. Dans cette approche, les objets fondamentaux sont les opérateurs de traces sur les infinis futur et passé qui associent à la solution son champ de radiation : l'opérateur de scattering est construit à l'aide de ces opérateurs de trace. L'avantage des ces techniques est qu'elles sont a priori indifférentes à la dépendance en temps de la géométrie. Mason-Nicolas les ont adaptées pour construire des théories de scattering conformes pour les champs de Dirac, Maxwell et l'équation des ondes sur les espaces-temps de Corvino-Schoen et Chrusciel-Delay. Ces travaux furent par la suite prolongés en un résultat de peeling complet dans l'espace de Schwarzchild ([63]).

Nous nous proposons ici d'utiliser des techniques similaires pour établir un résultat de scattering pour l'équation des ondes non linéaire conformément invariante défocalisante suivante :

$$\nabla_a \nabla^a \phi + b\phi^3 = 0$$

sur des espaces temps asymptotiquement simples de type Chrusciel-Delay et Corvino-Schoen. Le problème de Cauchy pour cette équation non linéaire a été étudié par Cagnac–Choquet-Bruhat dans [11]. Notre construction repose sur des méthode de champs de vecteurs, c'est-à-dire sur des estimations d'énergies géométriques. La stratégie de la preuve est donc la suivante : on se donne une surface de Cauchy Σ_0 et l'on s'intéresse à l'équation des ondes conforme sur le compactifié de l'espace-temps. Un champ de vecteurs uniformément temporel sur la variété compactifiée est choisi. La preuve de l'existence d'un opérateur de scattering est alors établie en trois étapes, qui sont détaillées ci-après :

1. des estimations d'énergies a priori sont établies pour une solution sur l'espace-temps compactifié ;
2. on montre ensuite que le problème caractéristique posé sur le cône de lumière $\mathcal{I}^+$ est bien posé pour de petites données ;
3. les opérateurs de trace, puis l'opérateur de scattering conforme, sont alors construits.

Nous détaillons maintenant ces trois étapes. Dans la première partie, nous montrons que la norme H^1 de la trace de la solution sur $\mathcal{I}^+$ est contrôlées par la norme des données sur Σ_0 et réciproquement. Les principaux obstacles sont ici les singularités en i^0 et i^+. Les estimations d'énergies sont établies spécifiquement dans un voisinage de chacune de ces singularités. Pour contourner la difficulté due à la singularité en i^0, le choix du vecteur utilisé est ajusté : suivant les travaux de Mason–Nicolas, nous choisissons un champ de vecteurs coïncidant avec le champ de vecteurs de Morawetz dans un voisinage de i^0. Ce champ a été utilisé dans [66] pour obtenir des estimations de décroissance ponctuelle d'une solution de l'équation des ondes en espace-temps plat. Il est adapté dans le cas de la métrique de Schwarzschild dans [21] par Dafermos-Rodnianski puis par Mason-Nicolas dans [63]. La singularité en i^+ est traitée en suivant la technique de Hörmander dans [53] : l'infini conforme est décrit comme le graphe d'une fonction. Il est important de noter que, dans cette partie, les hypothèses faites sur la fonction b, permettent de travailler avec une énergie contenant un terme non-linéaire. Le problème des injections de Sobolev uniformes n'apparaît donc qu'à la partie suivante.

La seconde partie est dédiée au problème de Cauchy sur l'hypersurface caractéristique $\mathcal{I}^+$. L'une des principales difficultés pour établir l'existence et l'unicité des solutions est d'obtenir des estimations d'énergie pour le propagateur associé à l'équation des ondes. Il est necessaire pour obtenir de telles estimations en contrôlant la non linéarité d'établir un résultat d'injections de Sobolev uniformes sur un feuilletage. La stratégie utilisée est la suivante : considérant un feuilletage régulier par des hypersurfaces uniformément spatiales, les feuillets sont étendus à $\mathbb{R}^3$ à l'aide d'un opérateur de prolongement dont la norme est contrôlée par la constante de Lispschitz associée aux bords des feuillets (voir le résultat de Stein dans [81]). Il est alors possible d'utiliser les résultats de Hébey ([51]) pour contrôler uniformément la norme des injections de Sobolev en fonction de la géométrie du feuilletage. Utilisant les estimations obtenues sur le propagateur, on construit par approximation une solution au problème de Goursat à données petites dans un voisinage de $\mathcal{I}^+$ par approximation par des fonctions solution d'un problème de Goursat linéaire avec source. Le théorème d'existence établi par Cagnac–Choquet-Bruhat ([11]) permet alors d'étendre la solution à tout l'espace-temps en résolvant un problème de Cauchy.

Finalement, on construit dans une dernière partie l'opérateur de scattering conforme Lipschitz à partir des opérateurs de trace. Cette construction découle immédiatement des parties précédentes.

Il nous a enfin semblé intéressant dans une partie additionnelle de présenter une stratégie « plus naturelle » du point de vue du scattering. On combine l'usage des méthodes conformes avec un choix de champs de vecteurs provenant d'une fonction de temps sur l'espace-temps physique. Un tel champ est tangent à $\mathcal{I}^+$, ce qui donne à l'infini des nomes plus faibles que celles obetnues en utilisant un champ temporel transverse à $\mathcal{I}^+$. Cette méthode produit bien les estimations a priori attendues dans une voisinage de i^0. Néanmoins, le contrôle de la forme de Killing de ce champ de vecteurs dans un voisinage de i^+ est un problème difficile. C'est pourquoi il a été choisi de se tourner vers le gradient du facteur conforme. Ce champ de vecteurs, largement étudié par Penrose ([79]) à des fins de peeling, a été utilisé avec succès par Mason-Nicolas dans [62] pour obtenir des estimations pour l'équation de Dirac au voisinage de i^+. Cette étude n'a pas abouti à l'heure où ce travail de thèse se termine. Cette partie est une ébauche et sera développée dans un travail à venir.

Je tiens à remercier Christian Gérard, qui a accepté de présider le jury examinant ma thèse, les rapporteurs, Lars Andersson et Philippe LeFloch, pour le temps qu'ils ont consacré à son évaluation, et les membres du jury, Dietrich Häfner, Stephen Huggett et Rachid Regbaoui.

J'adresse tout particulièrement mes remerciements à mon directeur de thèse, Jean-Philippe Nicolas, qui a tenu bon contre la mauvaise foi, le déni, le doute, les excès, la colère, bref, contre mes nombreuses qualités. Pis, Jean-Philippe a eu la bonté de me donner un peu plus que des mathématiques en partageant aussi sa culture et sa bonne humeur. Je lui en suis grandement reconnaissant et redevable.

Nombreux furent ceux qui, durant la préparation cette thèse à Brest, m'ont, eux aussi, soutenu, secoué et supporté. Qu'ils soient donc eux aussi remercier : Laurent, Jérôme, Chloé, Youna, Aurélien, Michael, Mickael, Gaelle, Nicolas, Nolwenn, Anne-Lise.

Du point de vue du doctorant, si certes tout n'est pas parfait, il a régné une bonne humeur sur le laboratoire de mathématiques de Brest. A commencé dans les bureaux des thésards. La vingtaine de kilogrammes (si, si) de pâte au chocolat à tartiner qui nous avons mangé ensemble au thé y est certainement pour quelque chose. Et quelques repas et sorties partagés aussi. Merci donc à Matteo, Christine, Jean-Baptiste, Qian et Shuai.

Il faut aussi dire que tous les membres de ce laboratoire ont été accueillants et disponibles. Je tiens à remercier spécifiquement Rachid Regbaoui et Pierre Cardaliaguet qui ont enduré mes questions et y ont répondu utilement.

Je tiens enfin à remercier Dietrich Häfner qui a participé activement au financement de cette thèse, via le financement ANR JC0546063 "Équations hyperboliques dans les espaces-temps de la relativité générale".

Il serait enfin injuste que je ne remercie pas ici Jean-Luc et son équipage, ainsi que le docteur, avec lesquels j'ai passé beaucoup de temps.

Sommaire

Chapter 1

Integral formula for the characteristic Cauchy problem for the Dirac equation on a curved background

Abstract: We give a local integral formula, valid on general curved space-times, for the characteristic Cauchy problem for the Dirac equation with arbitrary spin using the method developed by Friedlander in [39]. The results obtained by Penrose in the flat case in [78] are recovered directly. It is expected that this method can be used to obtain sharp estimates for the characteristic Cauchy problem for the Dirac equation.
Résumé: Nous donnons une formule intégrale pour le problème de Cauchy caractéristique local pour l'équation de Dirac pour le spin arbitraire en utilisant la méthode développée par Friedlander dans [39]. Nous retrouvons alors directement le résultat de Penrose dans le cas plat ([78]).

Introduction

Penrose obtained in 1963 ([78]) an integral formula for the characteristic Cauchy problem for the Dirac equations for arbitrary spin in the flat case. His derivation of the integral formula is based on the construction of a Newman-Penrose tetrad (null tetrad) adapted to the null structure of the null initial data hypersurface, and especially to the description of the behavior of the null generators (bicharacteristics) of the cone. The use of the 2-spinor formalism allows him to write the solution of the problem in function of a "null datum", contraction of the data on the cone with its spinor generators. The formula is verified a posteriori through a splitting of the Dirac operator over the spin basis in the compacted spin coefficient formalism. Penrose expected that this formula could be extended to the analytic case. As far as the author knows, the general case remains open.

Friedlander gave in the mid 70's ([39]) a method to obtain a parametrix for the wave equation derived from the Leray constructions (see for instance [43]) and wrote an integral representation of the solution of the characteristic Cauchy problem. His construction is based on a natural decomposition of the fundamental solution on the cone. Another approach exists to the characteristic Cauchy problem based on Fourier Integral operators. It must furthermore be noticed that there is no general result about the characteristic Cauchy problem for hyperbolic operators. Hörmander gave in [53] a general result of existence and uniqueness, together with energy estimates, for the wave equation on a spatially compact Lorentzian manifold.

The purpose of this paper is to combine the method developed by Friedlander with the description of the null cone by Penrose to obtain an integral formula for the characteristic Cauchy problem with initial data on the cone for arbitrary spin in general curved space-times. The choice of this method implies that we face the same restrictions as in the book by Friedlander. There exists an essential obstacle to the extension of the domain of validity of the representation formula: the existence of caustics which limit the domain where the formula can be written. We then have to restrict ourselves to a geodesically convex domain Ω of a smooth Lorentzian manifold (M, g), that is to say a domain where there exists a unique geodesic between any pair of distinct points. This restriction is inherent to the method and the fact we work with arbitrary curved geometry. The advantage is however that we obtain an explicit integral formula without resorting to any microlocalization. This in principle should allow an extension to metrics of low regularity in the spirit of [60].

More explicitly, let us consider (M, g) a smooth Lorentzian manifold and p_0 a point in Ω; the problem:

$$\not{D}u = 0$$

where u is a section of a given fiber bundle on Ω and $\not{D}$ is the Dirac operator on this bundle, with the initial conditions on the future null cone $\mathcal{C}^+(p_0)$:

$$u = \theta \text{ on } \mathcal{C}^+(p_0) \cap \Omega$$

is known as a first order Goursat problem with initial data on the characteristic hypersurface $\mathcal{C}^+(p_0) \cap \Omega$.

It is known that several conditions must be satisfied to ensure that this problem admits a solution. The first one comes from a geometric obstruction to the existence of a solution when symmetry conditions on the field u are imposed; this implies that the manifold M

must satisfy some geometric assumptions, known as the consistency conditions, depending on the spin we are working with. The second one comes from the fact that the initial data are given on a characteristic hypersurface: θ must then satisfy the restriction of the Dirac equation to the cone from p_0:

$$\not{D}|_{\mathcal{C}^+(p_0)}\theta = 0.$$

These equations are called the compatibility equations for the initial data.

As already mentioned, there exists, as far as the author knows, no general result about the characteristic Cauchy problem. Nonetheless, it is worth mentioning some results of existence and uniqueness with some generality. In the analytical case, this problem is similar to the Cauchy-Kowaleski problem (see for instance [43]). The problem is well posed in that case. This can be extended, with energy estimates, to minimal regularity ([49]). The well-posedness of the characteristic Cauchy problem is nonetheless not the point of this paper: assuming existence and uniqueness of the solution in the neighborhood of the point p_0, the goal consists in deriving a representation formula for this solution.

The paper is organised as follows. The first part presents an adaptation of the Friedlander method to the bundle of Dirac spinors. After a geometric and intrinsic presentation of the theory of spinors, the analytic tools to write a fundamental solution of the Dirac equation are developed. The second part is devoted to the derivation of the formula for Dirac spinors. Following Penrose's construction, a null tetrad adapted to the structure of the null cone is constructed and used to describe the geometric tools. The integral formula can then be derived from the parametrix and the result obtained by Penrose is recovered for Weyl (or two-) spinors. Finally, the third part deals with the arbitrary spin $\frac{n}{2}$. The presentation made in the first part is adapted to the bundle of spinors with spin $\frac{n}{2}$ so that the construction can be applied directly. A representation formula is then given for arbitrary spin and simplified in the case of the Maxwell equations. Penrose's formula for the characteristic Cauchy problem for arbitrary spin in the flat case is recovered in a flat spacetime.

Notations and conventions.

We describe here for future reference the notations and conventions which will be used all along the paper. Note that smooth means C^∞ in this paper.

1. Geometric notations:

   (a) General framework:

   - (M, g): smooth Lorentzian oriented and time oriented manifold with a metric g having signature signature $(+, -, -, -)$;
   - Ω: geodesically convex domain of M;
   - μ: volume form associated with the metric g on M;
   - p_0: a given point in Ω;
   - ∇: Levi-civita connection for g on the tangent bundle of M, TM.

   (b) Null structure on Ω: let p be a given point in Ω:

   - $\mathcal{C}(p)$: null cone from p, that is to say the set of points of Ω which lie on a null geodesic passing through p;
   - $\mathcal{C}^+(p)$ (resp. $\mathcal{C}^-(p)$): future (resp. past) null cone from p, that is to say the set of points of Ω that lie on a future (resp. past) oriented null geodesic from p;

- $\mathcal{I}(p)$: chronological set from p, that is to say the points of Ω which lie on a timelike or null geodesic passing through p;
- $\mathcal{I}^+(p)$ (resp. $\mathcal{I}^-(p)$): future (resp. past) chronological set from p, that is to say the set of points of Ω that lie on a future (resp. past) oriented timelike or null geodesic from p;
- $\mathcal{J}(p) = \mathcal{I}(p)\backslash\mathcal{C}(p)$: causal set from p and $\mathcal{J}^{\pm}(p) = \mathcal{I}^{\pm}(p)\backslash\mathcal{C}^{\pm}(p)$ are the future and past causal sets from p.

(c) Spin structure: Ω is endowed with a spin structure; the spinors will be denoted using the Penrose conventions as well as the usual algebraic notations according to convenience:

- $\mathbb{S}_{Dirac}$: fibre bundle of Dirac (or 4-) spinors;
- $\mathbb{S}_A$ and $\mathbb{S}^{A'}$: bundles of Weyl (or 2-) spinors (resp. dual and anti-spinors);
- "$\cdot$": Clifford multiplication;
- $(\cdot,\cdot)$: symplectic product on $\mathbb{S}_{Dirac}$ obtained by lifting the metric g;
- ϵ^{AB} and $\epsilon_{A'B'}$: restrictions of $(\cdot,\cdot)$ to $\mathbb{S}_A$ and $\mathbb{S}^{A'}$;
- $\mathbb{C}_0^\infty(\mathbb{S}_{Dirac}) = \mathcal{D}(\mathbb{S}_{Dirac})$: smooth sections with compact support in Ω endowed with the usual Fréchet topology;
- $\mathcal{D}'(\mathbb{S}_{Dirac})$: its topological dual;
- $\mathbb{C}^\infty(\mathbb{S}_{Dirac}) = \mathcal{E}(\mathbb{S}_{Dirac})$: smooth sections of $\mathbb{S}_{Dirac}$ on Ω,;
- $\mathcal{E}'(\mathbb{S}_{Dirac})$: its topological dual;
- the connection ∇ on $T\Omega$ is lifted on $\mathbb{S}_{Dirac}$ and is still denoted ∇;
- the Dirac operator is defined, for a given section $(e_i)_{i\in\{0,\dots,3\}}$ of the fibre bundle of orthonormal frames, on $C^\infty(\mathbb{S}_{Dirac})$ by:

$$\forall\Phi \in C^\infty(\mathbb{S}_{Dirac}), \not{D}\Phi = \sum_{i\in\{0,\dots,3\}} e_i \cdot \nabla_{e_i}\Phi$$

1 Geometric and analytic preliminaries

The geometric and analytic tools are presented in this section. As already mentioned in the introduction, due to geometric obstructions such as conjugates points or convergence of geodesics, the whole paper restricts itself to a geodesically convex domain Ω:

Definition 1.1. *A domain Ω is said to be geodesically convex if and only if it is an open set where, for every pair of points (p, q) in Ω, there exists a unique geodesic between p and q.*

1.1 Dirac spinors and Dirac equation

This section presents a construction of the spinor bundle so that it will be possible to apply the method of Friedlander in the most direct way. This presentation also intends to be a small dictionary between an abstract presentation of the theory of spinors and the Penrose conventions to represent spinors in terms of indices. Finally it must be noticed that, though the presentation is made on Ω, it can be generalized to a globally hyperbolic manifold (see remark 1.3 below).

1.1.1 Abstract construction

We begin by defining a spin bundle:

Definition 1.2. *A manifold M is said to be spin if its tangent bundle admits a spin structure, that is to say there exists a Spin(1,3) principal bundle $P_{\mathbb{S}}$, together with a twofold covering $\xi : P_{\mathbb{S}} \to P_{SO}M$, where $P_{SO}M$ is the $SO(1,3)$-principle bundle of orthonormal frames on M, such that*

$$\forall p \in P_{\mathbb{S}}, \forall g \in Spin(1,3), \xi(pg) = \xi(p)\xi_0(g).$$

where ξ_0 is the universal covering from $Spin(1,3) \approx SL_2(\mathbb{C})$ on $SO(1,3)$.

Remark 1.3.

1. *The existence of a spin structure on a manifold is usually ensured by the assumption that its second Stiefel Whitney class vanishes.*
2. *In the case of a four dimensional Lorentzian manifold (M, g), Geroch showed in [44] that a necessary and sufficient condition for M to carry a spin structure is that its bundle of orthonormal frames admits a global section (this is referred to as parallelizability).*
3. *A common assumption in general relativity which ensures that a 4-dimensional Lorentzian manifold is spin is the global hyperbolicity assumption: there exists in M a global Cauchy hypersurface, i.e. a spacelike hypersurface such that any inextendible timelike geodesic intersects this surface exactly once([44, 45]).*

The spinor bundle on Ω is defined through the action of an algebra over a vector space. This construction requires the following tool, which consists in group action over a fibre bundle, replacing its previous fibre by a given vector space:

Definition 1.4. *Let (E, Ω, π) be a G-principal bundle. Let F be a vector space and $\rho : G \to Homeo(F)$ a continuous map.*
Consider the action

$$\begin{array}{rccc} \phi : & G & \to & Aut(E) \times Homeo(F) \\ & g & \mapsto & \big((x, y) \in E \times F \mapsto (xg^{-1}, \rho(g)y)\big) \end{array} .$$

The quotient space

$$E \times F/\rho \text{ or } E \times_\rho F$$

with projection $\tilde{\pi}$ obtained by factorization of the diagram

$$\begin{tikzcd} E \times F \arrow[rr, "\pi \circ p"] \arrow[dr, "\phi"'] & & M \\ & E \times_\rho F \arrow[ur, "\tilde{\pi}"'] & \end{tikzcd}$$

where $p : E \times F \to F$ is the projection on the first variable, is a G-principal bundle with fibre F.

The algebra, known as the Clifford algebra associated with a given quadratic form, which is used to construct the spinor bundle is then defined:

Definition 1.5. *Let E be a vector space (real or complex) with a quadratic form q. The Clifford algebra $(Cl(E), +, \cdot)$ is the quotient space:*

$$Cl(E,q) = \left(\bigoplus_{n=0}^{+\infty} \bigotimes^{n} E \right) / I(E)$$

where $I(E)$ is the ideal generated by the set $\{v \otimes v - q(v) | v \in E\}$.

This algebra is known to have the following structure ([61]):

Proposition 1.6. *There exist two sub-algebrae denoted $Cl^0(E,q)$ and $Cl^1(E,q)$ such that:*

$$Cl(E,q) = Cl^0(E,q) \oplus Cl^1(E,q)$$

wich satisfy:

$$\begin{array}{ll} Cl^0(E,q) \cdot Cl^0(E,q) = Cl^0(E,q), & Cl^1(E,q) \cdot Cl^1(E,q) = Cl^0(E,q) \\ Cl^0(E,q) \cdot Cl^1(E,q) = Cl^1(E,q), & Cl^1(E,q) \cdot Cl^0(E,q) = Cl^1(E,q) \end{array} \tag{1.1}$$

Definition 1.7. *The group $Spin(E,q)$ is the subset of $Cl^0(E,q)$ defined by*

$$\{s \in Cl^0(E,q) | q(s) = 1\}$$

where q is the extension of the quadratic form q to $Cl(E,q)$.

The formalism previously defined can of course be applied to the case of the Minkowski spacetime $(\mathbb{R}^4, \eta)$.

Definition 1.8. *The bundle defined by:*

$$\mathbb{S}_{Dirac} = (P_{\mathbb{S}} \times M_2(\mathbb{C})) / (Spin(1,3))$$

is called the bundle of four dimensional spinors or bundle of Dirac spinors.

Remark 1.9.

1. *The representation of $Spin(1,3) = SL_2(\mathbb{C})$ acting on $M_2(\mathbb{C})$ has two irreducible components, which correspond to $\mathbb{C}^2$ with its two inequivalent complex structures; by convention, we write:*

$$M_2(\mathbb{C}) = (\mathbb{C}^2)^* \oplus \overline{\mathbb{C}^2}.$$

2. *The previous remark gives a decompositon of the fibre bundle $\mathbb{S}_{Dirac}$ into two bundles (known as bundles of Weyl spinors), corresponding to the splitting of $M_2(\mathbb{C})$ into $\overline{\mathbb{C}}^2$ and $\left(\mathbb{C}^2\right)^*$; this decomposition is written in terms of indices:*

$$\mathbb{S}_{Dirac} = \mathbb{S}_A \oplus \mathbb{S}^{A'}.$$

A section u of the $\mathbb{S}_{Dirac}$ bundle will then be split into two smooth sections of the Weyl bundles:

$$u = \phi_A \oplus \psi^{A'}.$$

3. *To get back to the tangent bundle, a convention must be chosen to represent the Clifford algebra $Cl(\mathbb{R}^4, \eta)$. Its usual representation is $M_2(\mathbb{H})$, which is split in $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$. The vectors are identified with the hermitian 2-forms or with $\mathbb{C}$-antilinear homomorphisms from $\mathbb{S}_A$ to $\mathbb{S}^{A'}$. As such, a vector u^a will be written by convention $u^{AA'}$.*

4. *For the chosen representation of the Clifford algebra which was made previously, the tangent bundle is identified to the set of hermitian two forms over $\mathbb{S}_A$. As such, it endows the tangent bundle with a structure of conformal Lorentzian manifold, i.e. a fibre bundle of cones over M and a time orientation: the fibre bundle of cones over Ω is made of degenerate hermitian two-forms, the spacelike vectors fields are the hermitian matrices of signature (2,0) or (0,2) and the timelike vectors fields are the hermitian matrices of signature (1,1). The time orientation is obtained by a choice of orientation on the fibre bundle of cones over Ω.*

Proposition 1.10. *The bundle $\Lambda^2\mathbb{S}^{A'}$ of skew-symmetric two forms is trivial.*

Proof. This is a direct consequence of the fact that $\mathbb{S}^{A'}$ is a $Spin(1,3) = SL_2(\mathbb{C})$ bundle. Let $(U \times \overline{\mathbb{C}^2}, \phi)$ and $(V \times \overline{\mathbb{C}^2}, \psi)$ be two local trivializations of the bundle $\mathbb{S}^{A'}$ with empty intersection where ϕ and ψ satisfy

$$p \circ \phi = \pi \text{ and } p \circ \psi = \pi$$

where $\pi : \mathbb{S}^{A'} \to M$ is the projection associated to the bundle $\mathbb{S}^{A'}$ and p is the projection on the first variable. These two trivializations give rise to two trivializations of $\Lambda^2\mathbb{S}^A$ that are still denoted by ϕ and ψ. Let us consider the transition map $\phi\circ\psi^{-1} : \Lambda^2\mathbb{C}^2\times V \to \Lambda^2\mathbb{C}^2\times U$. It can be written:

$$\phi \circ \psi(x, y) = (x, \nu(x)y)$$

where $\nu : U \cap V \to SL_2(\mathbb{C})$ is a smooth map.
Let x be fixed in $U \cap V$. Since $\nu(x)$ belongs to $SL_2(\mathbb{C})$ and y is a skew-symmetric 2-form, y is invariant under the action of an element of $SL_2(\mathbb{C})$, i.e:

$$\forall (u, v) \in \mathbb{C}^2, y(\nu(x)u, \nu(x)v) = y(u, v).$$

The fibre bundle $\Lambda^2\mathbb{S}^{A'}$ is thus trivial.♦

Remark 1.11. 1. *The canonical isomorphism, which will be denoted by κ, between $\mathbb{S}^{A'}$ and $\mathbb{S}_A$ induces an other isomorphism between $\Lambda^2\mathbb{S}^{A'}$ and $\Lambda^2\mathbb{S}_A$:*

$$\begin{array}{ccl} \Lambda^2\mathbb{S}^{A'} & \longrightarrow & \Lambda^2\mathbb{S}_A \\ \epsilon & \longmapsto & \kappa_*\epsilon : (u, v) \in \mathbb{S}_A \times \mathbb{S}_A \mapsto \epsilon(\kappa(u), \kappa(v)). \end{array}$$

It allows the construction of a symplectic form on $\mathbb{S}_{Dirac}$: let ϵ in $\Lambda^2\mathbb{S}^{A'}$, we obtain a symplectic form on $\mathbb{S}_{Dirac}$ by taking:

$$\epsilon \oplus \kappa^*\epsilon.$$

A two-form ε on $\mathbb{S}_A$ is denoted ε^{AB} and acts on Weyl spinors by:

$$\forall (u_A, v_B) \in \mathbb{S}_A, \varepsilon(u, v) = \varepsilon^{AB}u_A v_B.$$

The corresponding two-form on $\mathbb{S}^{A'}$ is denoted $\varepsilon_{A'B'}$.

2. *Let ε be a fixed skew-symmetric two-form on $\mathbb{S}_A$. It is possible to construct a metric $\tilde{g}$ on $T\Omega$ by, for $x^{AA'}$ and $y^{AA'}$ two vectors:*

$$g_{ab}u^a v^b = \varepsilon_{AB}\varepsilon_{A'B'}x^{AA'}y^{BB'}$$

3. *We denote by ε_{AB} a two form which gives rise to the metric g on M. The non-degeneracy of ε induces an indentification between $\mathbb{S}_A$ and its dual $\mathbb{S}^A$ given by:*

$$\kappa_A \in \mathbb{S}_A \longmapsto \kappa^A = \varepsilon^{AB}\kappa_B \in \mathbb{S}^A$$

whose inverse mapping is

$$\kappa^B \in \mathbb{S}^B \longmapsto \kappa_B = \kappa^A\varepsilon_{AB}.$$

The equivalent transformation can be made for the complex conjugate spinors in $\mathbb{S}^{A'}$ if we consider the image two-form $\varepsilon_{A'B'}$.

4. *The symplectic product on Dirac spinors can thus be written, by lowering and raising indices:*

$$\begin{aligned}(u,v) &= \varepsilon^{AB}\psi_A\phi_B + \varepsilon_{A'B'}\xi^{A'}\zeta^{B'} \\ &= -\psi^A\phi_A + \xi_{A'}\zeta^{A'}\end{aligned}$$

where $u = \psi_A + \xi^{A'}$ and $v = \phi_A + \zeta^{A'}$ are two Dirac spinors.

5. *The dual $\mathbb{S}^\star_{Dirac}$ of $\mathbb{S}_{Dirac}$ is split in:*

$$\mathbb{S}^\star_{Dirac} = \mathbb{S}_{A'} \oplus \mathbb{S}^A.$$

The symplectic form $(\cdot,\cdot)$ realizes an identification between $\mathbb{S}_{Dirac}$ and $\mathbb{S}^\star_{Dirac}$, whereas its restrictions to, respectively, $\mathbb{S}_A$ and $\mathbb{S}^{A'}$, denoted ε^{AB} and $\varepsilon_{A'B'}$, realize an identification between $\mathbb{S}^A$ and $\mathbb{S}_A$ and between $\mathbb{S}_{A'}$ and $\mathbb{S}^{A'}$ respectively.

Proposition 1.12. *Let ε be a section of $\Lambda^2\mathbb{S}^A$. Let $\tilde{g}$ be the metric associated with ε. Then, the metric $\tilde{g}$ is conformal to the metric g.*

Proof. : Let p in M. Let X in T_pM and u, v in $\mathbb{S}_A$. We assume that the vector X is a light-like vector for the metric g, i.e.:

$$g(X,X) = g_{ab}X^aX^b = 0.$$

A necessary and sufficient condition for $\tilde{g}_{ab}$ to be conformal to g is that $\tilde{g}$ and g have the same null cone structure, i.e. it is sufficient to show:

$$\tilde{g}(X,X) = \tilde{g}_{ab}X^aX^b = 0.$$

Since X^a is light-like vector, it can be written:

$$X^a = u^A\overline{u}^{A'}.$$

if it is future directed and

$$X^a = -u^A\overline{u}^{A'}$$

if it is past directed. The calculation is performed for a future directed null vectors, but it is the same for a past directed one. Because of the skew-symmetry of ε_{AB}, we have:

$$\varepsilon_{AB}u^Au^B = 0$$

and then

$$\tilde{g}_{ab}X^aX^b = \varepsilon_{AB}u^Au^B\varepsilon_{A'B'}\overline{u}^{A'}\overline{u}^{B'} = 0.$$

X is thus still a null vector for $\tilde{g}$ and $\tilde{g}$ and g are conformal metrics. ♦

Remark 1.13. *The map:*

$$\begin{array}{ccc} \Lambda^2\mathbb{S}^A & \longrightarrow & \{\phi g | \phi \in C^\infty(\Omega, \mathbb{R}_+^*)\} \\ \varepsilon_{AB} & \longmapsto & g_{ab} = \varepsilon_{AB}\varepsilon_{A'B'} \end{array}$$

is a two sheeted covering of the conformal class of g. In particular, it is surjective. We denote by ε_{AB} a preimage of g_{ab}.

Proposition 1.14. *The bundle $\mathbb{S}_{Dirac}$ is a Dirac bundle, i.e. a fibre bundle of left modules over $Cl(\Omega, g)$ endowed with a symplectic form ϵ and a connection $\nabla^{\mathbb{S}}$ such that:*

1. *$\nabla^{\mathbb{S}}$ is the pull-back of the Levi-Civita connection on M: if $\pi : \mathbb{S}_{Dirac} \to \Omega$, then ∇ can be written:*
$$\pi^*\nabla = \nabla^{\mathbb{S}}$$

2. *the connection is compatible with the action of the Clifford algebra: let X be a smooth section of $Cl(T\Omega, g)$ and u a smooth section of $\mathbb{S}_{Dirac}$, then:*
$$\nabla^{\mathbb{S}}(X \cdot u) = \nabla X \cdot u + X \cdot \nabla^{\mathbb{S}} u.$$
Though different since they are acting on different objects, the connexion ∇ on Ω and $\nabla^{\mathbb{S}}$ on $\mathbb{S}_{Dirac}$ are both denoted by ∇.

3. *the action of the Clifford multiplication is an isometry for the symplectic product: let X be a smooth section of $Cl(T\Omega, g)$ and u, v two smooth sections of $\mathbb{S}_{Dirac}$, then:*
$$\epsilon(X \cdot u, X \cdot v) = q(X)\epsilon(u, v).$$

In order to define the space on square integrable spinors on Ω, it is necessary to define the norm of a spinor. This unfortunately cannot be done without choosing a time function t (see [71]) or, at least, a timelike vector field.

Definition 1.15. *A smooth function t on Ω is called a time function if, and only if its gradient is a non-vanishing future-oriented timelike vector field on U.*

Definition 1.16. *Let t be a time function on Ω. Then the map defined by:*

$$\begin{array}{ccc} C_0^\infty(\Omega, \mathbb{S}_{Dirac}) \times C_0^\infty(\Omega, \mathbb{S}_{Dirac}) & \longrightarrow & \mathbb{R} \\ (\Psi, \Phi) & \longmapsto & \varepsilon(\nabla t \cdot \Psi, \Phi) \end{array}$$

is a positive definitive hermitian product over the set of smooth sections of $\mathbb{S}_{Dirac}$ with compact support in U. The norm associated to this scalar product is denoted by $|| \star ||_U$.

Remark 1.17.

- *This norm will be used in the following on compact subsets of an open set of Ω to define the Fréchet topology over smooth sections of the fiber bundle of Dirac spinors.*
- *A time function t is fixed on Ω. This time function will be used to compute all the norms.*

This scalar product is used to define various norms over the spinor fields on Ω: let Φ in $\mathcal{D}(\mathbb{S}_{Dirac})$. We define using the positive definite hermitian product:

- the L^∞-norm over a compact K of Ω:

$$||\Phi||_{\infty,K} = \sqrt{\sup_K \left(\varepsilon(\nabla t \cdot \Phi, \Phi)\right)};$$

- if $\psi : \Omega \to \mathbb{R}^4$ is a given chart over Ω, the norm over K, for any integer N:

$$||\Phi||_{\infty,N,K} = \sqrt{\sum_{\substack{\alpha \\ |\alpha|\leq N}} \sup_K \left(\varepsilon(\nabla t \cdot \nabla^\alpha \Phi, \nabla^\alpha \Phi)\right)},$$

  where $\nabla^\alpha = \nabla_{\partial_{\alpha_1}} \nabla_{\partial_{\alpha_2}} \dots \nabla_{\partial_{\alpha_l}}$, $\alpha = (\alpha_1, \dots, \alpha_l)$ being a multi-index of length $|\alpha| = \sum_{i=1\dots n} \alpha_i$; a chart of reference Ψ is fixed in the following in the computation of the norms;

- the L^2-norm over Ω:

$$||\Phi||_2 = \sqrt{\int_\Omega \varepsilon(\nabla t \cdot \Phi, \Phi)\mu}.$$

1.1.2 Newman-Penrose tetrad.

One way to describe the Lorenztian structure is to use a global section of the fiber bundle of orthonormal frames over Ω and translate the result in terms of spinors. We construct a global basis, named tetrad of Newman-Penrose which gives rise to a spinor basis of $\mathbb{S}^A$.

Definition 1.18. *A basis of $T\Omega \otimes \mathbb{C}$ $(l, n, m, \overline{m})$ is called a normalized Newman Penrose basis if l and n are real vectors fields and it satisfies the following relations:*

$$\begin{array}{ccccccc} g(l,l) = 0 & , & g(n,n) = 0 & , & g(m,m) = 0, & & \\ g(l,n) = 1 & , & g(m,\overline{m}) = -1 & , & g(l,m) = 0 & , & g(n,m) = 0. \end{array}$$

Remark 1.19.
- *The existence of a Newman-Penrose tetrad is insured by the existence of a global section of the fibre bundle of orthonormal frames: if $(e^a_{\mathbf{a}})$ $(\mathbf{a} = 0,1,2,3)$ is such a section, the following family of vectors:*

$$\begin{array}{rclrcl} l^a &=& \frac{1}{\sqrt{2}}(e^a_0 + e^a_1) & m^a &=& \frac{1}{\sqrt{2}}(e^a_2 + ie^a_3) \\ n^a &=& \frac{1}{\sqrt{2}}(e^a_0 - e^a_1) & \overline{m}^a &=& \frac{1}{\sqrt{2}}(e^a_2 - ie^a_3) \end{array} \tag{1.2}$$

  is a normalized Newman-Penrose tetrad. It is obvious that a given normalized Newman-Penrose tetrad gives rise to an orthonormal basis of $T\Omega$ with the following reverse fomulae:

$$\begin{array}{rclrcl} e^a_0 &=& \frac{1}{\sqrt{2}}(l^a + n^a) & e^a_2 &=& \frac{1}{\sqrt{2}}(m^a + \overline{m}^a) \\ e^a_1 &=& \frac{1}{\sqrt{2}}(l^a - n^a) & e^a_3 &=& \frac{1}{i\sqrt{2}}(m^a - \overline{m}^a) \end{array}$$

- *Because the structure of null cones will be considered later, we assume that a Newman-Penrose tetrad $(l, n, m, \overline{m})$ is given first and, in a second time, gives rise to an orthonormal basis $(e^a_{\mathbf{a}})$ $(\mathbf{a} = 0,1,2,3)$.*

- *Up to an overall sign, there exist two unique spinor fields in $\mathcal{E}(\mathbb{S}^A)$, denoted by o^A and ι^A such that:*

$$l^a = o^A \bar{o}^{A'}, n^a = \iota^A \bar{\iota}^{A'} \text{ and } m^a = o^A \bar{\iota}^{A'}.$$

These two spinors are chosen such that the following normalization is satisfied:

$$\varepsilon_{AB} o^A \iota^B = o_A \iota^A = 1$$

- *There exists an alternative notation for this spin basis, which is consistent with the duality property used to describe spinors. We note, in $\mathbb{S}^A$:*

$$\varepsilon_0^A = o^A \text{ and } \varepsilon_1^A = \iota^A.$$

We also introduce their dual spinors in $\mathbb{S}_A$ $(\varepsilon_A^0, \varepsilon_A^1)$ which satisfy:

$$\begin{array}{lcl} \varepsilon_A^0 \varepsilon_0^A = 1 & , & \varepsilon_A^1 \varepsilon_1^A = 1, \\ \varepsilon_A^0 \varepsilon_1^A = 0 & , & \varepsilon_A^1 \varepsilon_0^A = 0; \end{array}$$

they are:

$$\varepsilon_A^0 = -\iota_A \text{ and } \varepsilon_A^1 = o_A.$$

- *The vector $e_{\mathbf{a}}^a$ can be written in function of the metric as $g_{\mathbf{a}}^a$ for $\mathbf{a} = 0, \dots, 3$. The components of its spinor form $g_{\mathbf{a}}^{AA'}$, called the Infeld-van der Waerden, defined as:*

$$g_{\mathbf{a}}^{\mathbf{AA'}} = e_{\mathbf{a}}^a \varepsilon_A^{\mathbf{A}} \varepsilon_{A'}^{\mathbf{A'}}.$$

are the coefficients of the decomposition of $e_{\mathbf{a}}^a$ in the basis $(\varepsilon_0^A, \varepsilon_1^A)$:

$$e_{\mathbf{a}}^a = g_{\mathbf{a}}^{\mathbf{AA'}} \varepsilon_{\mathbf{A}}^A \varepsilon_{\mathbf{A'}}^{A'}.$$

It is then known ([71], section 3.1) that the Clifford multiplication of a Dirac spinor by the basis vectors can be written:

Lemma 1.20. *The Clifford multiplication of a Dirac spinor $\phi_A + \psi^{A'}$ by the vector $e_{\mathbf{a}}^a$ is given by:*

$$e_{\mathbf{a}}^a \cdot (\phi_A \oplus \psi^{A'}) = i\sqrt{2} g^{\mathbf{a}}{}_{AA'} \psi^{A'} \oplus -i\sqrt{2} g^{\mathbf{a}AA'} \phi_A$$

Remark 1.21. *: The Clifford multiplication can be interpreted as a contraction with the corresponding vector of the basis (up to a factor $\pm i\sqrt{2}$) by writing:*

$$\begin{array}{rcl} e_{\mathbf{a}}^a \cdot (\phi_A + \psi^{A'}) & = & i\sqrt{2} g^{\mathbf{ab}} g_{\mathbf{b}AA'} \psi^{A'} - i\sqrt{2} g^{\mathbf{ab}} g_{\mathbf{b}}^{AA'} \phi_A \\ & = & i\sqrt{2} g(e_{\mathbf{a}}, e_{\mathbf{a}}) g_{\mathbf{a}AA'} \psi^{A'} - i\sqrt{2} g(e_{\mathbf{a}}, e_{\mathbf{a}}) g_{\mathbf{a}}^{AA'} \phi_A \end{array}$$

As a consequence, the Clifford multiplication by the vector $l^{AA'}$ is the contraction with $n^{AA'}$ and conversely the Clifford multiplication by $n^{AA'}$ is the contraction by $l^{AA'}$ (up to a factor $\pm i\sqrt{2}$):

$$\begin{array}{rcl} l \cdot (\phi_A + \psi^{A'}) & = & i\sqrt{2}(n_{AA'} \psi^{A'} - n^{AA'} \phi_A) \\ n \cdot (\phi_A + \psi^{A'}) & = & i\sqrt{2}(l_{AA'} \psi^{A'} - l^{AA'} \phi_A) \end{array} \tag{1.3}$$

We conclude this section by giving the abstract index expression of the Dirac operator on 4-spinors ([71], section 3.1):

Lemma 1.22. *The Dirac operator is decomposed as follows:*

$$\not{D}(\phi_A + \psi^{A'}) = i\sqrt{2}(\nabla_{AA'} \psi^{A'} - \nabla^{AA'} \phi_A)$$

1.2 Analytic requirements

1.2.1 Distributions on spinors

The purpose is to write weak solutions for the Dirac equation. The theory of distributions must thus be adapted to ensure properties of symmetry for the Dirac operator and the Clifford multiplication so that the construction of Friedlander can be used with few adaptations.

1.2.1.1 Fundamental properties The basic elements needed in the next section are sketched here. Spinor-valued distributions are defined in [24] to construct fundamental solutions for the Dirac equation. They were also developed in [83] to construct a Fourier integral operator for the propagator of the Dirac equation.

Definition 1.23. *A distribution u on the set $\mathcal{D}(\mathbb{S}_{Dirac})$ of smooth Dirac spinor fields with compact support on Ω, endowed with its usual Fréchet topology, is a $\mathbb{C}$-linear continuous mapping from $\mathcal{D}(\mathbb{S}_{Dirac})$ to $\mathbb{C}$, i.e. a mapping which satisfies for all compact K in Ω, there exists a positive constant C and an integer m depending only on K such that:*

$$\forall \phi \in \mathcal{D}(\mathbb{S}), |u(\phi)| \leq C||\phi||_{\infty,m,K}$$

The set of distributions on M will be denoted by $\mathcal{D}'(\mathbb{S}_{Dirac})$ and the duality bracket by $<,>$.

Definition 1.24. *The support of a distribution u is the complement of the largest open subset O of Ω such that any smooth function ϕ with support in O satisfies:*

$$< u, \phi >= 0.$$

The set of compactly supported distributions is denoted $\mathcal{E}'(\mathbb{S}_{Dirac})$ and is the topological dual of $\mathcal{E}(\mathbb{S}_{Dirac})$, set of smooth sections of $\mathbb{S}_{Dirac}$ on Ω.

If u is a locally integrable section of $\mathbb{S}^*_{Dirac} = \mathbb{S}^A \oplus \mathbb{S}_{A'}$, which can be written $u = \xi^A + \eta_{A'}$, it defines a distribution by:

$$\forall \Phi \in \mathcal{D}(\mathbb{S}_{Dirac}), < u, \Phi >= \int_\Omega -\xi^A \phi_A + \eta_{A'} \psi^{A'} \mu.$$

where the smooth section Φ is split as: $\Phi = \phi_A + \psi^{A'}$.

We define now the action of the covariant derivative in a direction V and of the Dirac operator on distributions by:

Proposition 1.25. *Let u be an element of $\mathcal{D}'(\mathbb{S}_{Dirac})$ and V be a smooth section nowhere vanishing of $T\Omega$. The distributions $\nabla_V u$ and $\not{D} u$ are defined by:*

$$\begin{aligned}
\forall \phi \in \mathcal{D}(\mathbb{S}_{Dirac}), \quad < \nabla_V u, \phi >_{\mathcal{D}'(\mathbb{S}^\star_{Dirac}),\mathcal{D}(\mathbb{S}_{Dirac})} &= - < u, \nabla_V \phi >_{\mathcal{D}'(\mathbb{S}^\star_{Dirac}),\mathcal{D}(\mathbb{S}_{Dirac})} \\
\forall \phi \in \mathcal{D}(\mathbb{S}_{Dirac}), \quad < \not{D} u, \phi >_{\mathcal{D}'(\mathbb{S}^\star_{Dirac}),\mathcal{D}(\mathbb{S}_{Dirac})} &= - < u, \not{D} \phi >_{\mathcal{D}'(\mathbb{S}^\star_{Dirac}),\mathcal{D}(\mathbb{S}_{Dirac})}
\end{aligned}$$

These definitions agree with the Leibniz rule and the fact that the connexion is compatible with the symplectic product on spinors.

We also need to define the Clifford multiplication with a vector:

Proposition 1.26. *Let u be an element of $\mathcal{D}'(\mathbb{S}_{Dirac})$ and V a smooth section of $T\Omega$. We define the distribution $V\cdot u$ in $\mathcal{D}'(\mathbb{S}_{Dirac})$ by:*

$$\forall \phi \in \mathcal{D}(\mathbb{S}_{Dirac}), < V\cdot u, \phi >_{\mathcal{D}'(\mathbb{S}^\star_{Dirac}),\mathcal{D}(\mathbb{S}_{Dirac})} = < u, V\cdot \phi >_{\mathcal{D}'(\mathbb{S}^\star_{Dirac}),\mathcal{D}(\mathbb{S}_{Dirac})} .$$

Proof.
The representation of the Clifford multiplication is the same for the dual $\mathbb{S}^*_{Dirac}$. Consequently, if $u = \phi^{A'} + \chi_A$ is in $\mathbb{S}_{Dirac}$ and $v = \rho_{A'} + \theta^A$ is in $\mathbb{S}^*_{Dirac}$, then:

$$< v, e_{\mathbf{a}}\cdot u >_{\mathbb{S}^\star_{Dirac},\mathbb{S}_{Dirac}} = -i\sqrt{2}g^{\mathbf{a}}{}_{AA'}\chi^{A'}\theta^A - i\sqrt{2}g^{\mathbf{a}AA'}\phi_A\rho_{A'}.$$

We notice that this expression is symmetric in A and A' so that we can conclude:

$$< e_{\mathbf{a}}\cdot v, u >_{\mathbb{S}^\star_{Dirac},\mathbb{S}_{Dirac}} = < v, e_{\mathbf{a}}\cdot u >_{\mathbb{S}^\star_{Dirac},\mathbb{S}_{Dirac}} .\blacklozenge$$

Remark 1.27. *When a distribution on $\mathbb{S}_{Dirac}$ is represented by a function from Ω into $\mathbb{S}_{Dirac}$, the symplectic product $(\cdot,\cdot)$ on $\mathbb{S}_{Dirac}$ is used to apply the distribution on a section of $\mathbb{S}_{Dirac}$. The duality bracket will be in that case written $(\cdot,\cdot)_{\mathcal{D}'(\mathbb{S}_{Dirac}),\mathcal{D}(\mathbb{S}_{Dirac})}$.*

The previous results need to be checked since the definitions given in (1.25) do not work when the symplectic product (or the ε spinor) is used. We first need the following lemmata on the action of Clifford multiplication and the Dirac operator:

Lemma 1.28. *For any Φ and Ψ Dirac spinor fields on Ω and V a vector field on Ω, we have:*

$$(V\cdot\Phi, \Psi) = -(\Phi, V\cdot\Psi)$$

Proof. : It is sufficient to verify the result for an element $e_{\mathbf{a}}$ of the frame. We calculate $(e_{\mathbf{a}}\cdot\Phi,\Psi)$ in components.

$$(e_{\mathbf{a}}\cdot\Phi,\Psi) = -i\sqrt{2}\varepsilon_{A'B'}g^{\mathbf{a}AA'}\xi_A\theta^{B'} + i\sqrt{2}\varepsilon^{AB}g^{\mathbf{a}}{}_{AA'}\chi^{A'}\rho_B$$

with $\Phi = \xi_A + \chi^{A'}$ and $\Psi = \rho_A + \theta^{A'}$. Noticing that:

$$\varepsilon^{AB}g^{\mathbf{a}}{}_{AA'} = -g^{\mathbf{a}B}{}_{A'} = -\varepsilon_{B'A'}g^{\mathbf{a}BB'} = \varepsilon_{A'B'}g^{\mathbf{a}BB'}$$

we obtain:

$$(e_{\mathbf{a}}\cdot\Phi,\Psi) = -i\sqrt{2}\varepsilon^{AB}g^{\mathbf{a}}{}_{BB'}\xi_A\theta^{B'} + i\sqrt{2}\varepsilon_{A'B'}g^{\mathbf{a}BB'}\chi^{A'}\rho_B = -(\Phi, e_{\mathbf{a}}\cdot\Psi).\blacklozenge$$

In order to verify the symmetry of the Dirac operator for the symplectic product, we will establish the following lemma:

Lemma 1.29. *Let Φ and Ψ two spinor fields on Ω. Then we have:*

$$(\not{D}\Phi,\Psi) = (\Phi,\not{D}\Psi) - div(V).$$

where V is a complex vector field.

Proof. : The formula is proved at each point of Ω; let then p be a point in Ω. Let (f_i) be a orthonormal basis on Ω such that, for all i in $\{0,1,2,3\}$:

$$\nabla_{f_i} f_i = 0 \text{ at } p.$$

For this choice of basis, we have, at the point p:

$$\begin{aligned}
(\not{D}\Phi, \Psi) &= \sum_{i\in\{0,1,2,3\}} (f_i \cdot \nabla_{f_i}\Phi, \Psi) \\
&= -\sum_{i\in\{0,1,2,3\}} (\nabla_{f_i}\Phi, f_i \cdot \Psi) \\
&= -\sum_{i\in\{0,1,2,3\}} \left\{\nabla_{f_i}(\Phi, f_i \cdot \Psi) - (\Phi, \nabla_{f_i}(f_i \cdot \Psi))\right\} \text{ (,) being compatible with the connection.} \\
&= -\sum_{i\in\{0,1,2,3\}} \left\{\nabla_{f_i}(\Phi, f_i \cdot \Psi) - (\Phi, f_i \cdot \nabla_{f_i}\Psi)\right\} \text{ since at } p \ \nabla_{f_i} f_i = 0 \\
&= (\Phi, \not{D}\Psi) - \sum_{i\in\{0,1,2,3\}} \nabla_{f_i}(\Phi, f_i \cdot \Psi).
\end{aligned}$$

Introducing the complex vector field v defined, at p, by:

$$V = \sum_{i=0}^{3} \mathfrak{f}_i(\Phi, f_i \cdot \Psi) f_i$$

with $\mathfrak{f}_i =< f_i, f_i >$, we notice that

$$\sum_{i\in\{0,1,2,3\}} \nabla_{f_i}(\Phi, f_i \cdot \Psi)$$

is the divergence of V.
We present an alternative way to perform this calculation with abstract indices; the Dirac spinors Φ and Ψ are split on $\mathbb{S}_{Dirac}$:

$$\begin{aligned}
\Phi &= \phi_A \oplus \rho^{A'} \\
\Psi &= \psi_A \oplus \chi^{A'}.
\end{aligned}$$

We now lead the calculation in the usual way:

$$\begin{aligned}
\frac{1}{\sqrt{2}}(\not{D}\Phi, \Psi) &= \varepsilon^{AB}(i\nabla_{AA'}\rho^{A'})\psi_B + \varepsilon_{A'B'}(-i)(\nabla^{AA'}\phi_A)\chi^{B'} \\
&= \varepsilon^{AB} i\nabla_{AA'}(\rho^{A'}\psi_B) + \varepsilon_{A'B'}(-i\nabla^{AA'}(\phi_A\chi^{B'})) \\
&\quad -\varepsilon^{AB}\rho^{A'} i\nabla_{AA'}\psi_B - \varepsilon_{A'B'}\phi_A(-i\nabla^{AA'}\chi^{B'}) \\
&= i\nabla_{AA'}(\varepsilon^{AB}\rho^{A'}\psi_B) + (-i)\nabla^{AA'}(\varepsilon_{A'B'}\phi_A\chi^{B'}) \\
&\quad +\rho^{A'} i\nabla^{B}_{A'}\psi_B - \phi_A(-i)\nabla^{A}_{B'}(\chi^{B'}) \\
&= i\nabla_{AA'}(\rho^{A'}\psi^A) + i\nabla^{AA'}(\phi_A\chi_{A'}) \\
&\quad +\rho^{A'} i\nabla^{BB'}\varepsilon_{B'A'}\psi_B + i\phi_A\varepsilon^{AB}\nabla_{BB'}\chi^{B'} \\
&= i\nabla_{AA'}(\rho^{A'}\psi^A) + i\nabla^{AA'}(\phi_A\chi_{A'}) \\
&\quad -\varepsilon_{A'B'}\rho^{A'} i\nabla^{BB'}\psi_B + i\varepsilon^{AB}\phi_A\nabla_{BB'}\chi^{B'} \\
&= i\nabla_{AA'}(\rho^{A'}\psi^A) + i\nabla^{AA'}(\phi_A\chi_{A'}) + \frac{1}{\sqrt{2}}(\Phi, \not{D}\Psi).
\end{aligned}$$

It must be noticed that, in this new calculation, the remaining term can obviously be identified as a divergence.♦

Remark 1.30. *The vector field*

$$V = \sum_{i=0}^{3} \mathfrak{f}_i(\Phi, f_i \cdot \Psi) e_i \tag{1.4}$$

is encountered several times in the following. Though it is used to perform the calculation, it does not seem to be intrinsic. It is nonetheless easy to give a more intrinsic sense to this computation. Let us consider the complex 1-form ω on Ω:

$$\begin{array}{ccc} T\Omega \otimes \mathbb{C} & \longrightarrow & \mathbb{C} \\ h & \longmapsto & (\Psi, h \cdot \Phi) \end{array}$$

The dual vector of this 1-form is the vector (1.4). *The calculation can then be easily reinterpreted when noticing:*

$$d \star \omega = \left(\sum_{i \in \{0,1,2,3\}} \nabla_{f_i}(\Phi, f_i \cdot \Psi) \right) \mu,$$

$\star$ *being the Hodge dual and μ the volume form associated with the metric g.*

Definition 1.31. *Let u be in $\mathcal{D}'(\mathbb{S}_{Dirac})$, and X in $C^\infty T\Omega$. The applications defined by*

$$\Phi \in \mathcal{D}(\mathbb{S}_{Dirac}) \longmapsto -(u, X \cdot \Phi)_{\mathcal{D}'(\mathbb{S}_{Dirac}), \mathcal{D}(\mathbb{S}_{Dirac})}$$

and

$$\Phi \in \mathcal{D}(\mathbb{S}_{Dirac}) \longmapsto (u, \not{D}\Phi)_{\mathcal{D}'(\mathbb{S}_{Dirac}), \mathcal{D}(\mathbb{S}_{Dirac})}$$

are distributions, denoted respectively by $X \cdot u$ and $\not{D}u$.

Proof. : This is a straightforward consequence of the previous lemma and the Stokes theorem.♦

Remark 1.32. *: These definitions agree with the previous lemmata when u is in $\mathcal{D}(\mathbb{S}_{Dirac})$.*

From this point, all the distributions will be assumed to be represented via the symplectic product.

If f is in $\mathcal{D}'(\mathbb{R})$ and U is a smooth spinor field on Ω, we define the distribution fU by:

$$\forall \phi \in \mathcal{D}(\mathbb{S}_{Dirac}), (fU, \phi)_{\mathcal{D}'(\mathbb{S}_{Dirac}), \mathcal{D}(\mathbb{S}_{Dirac})} = < f, (U, \phi) >_{\mathcal{D}'(\mathbb{R}), \mathcal{D}(\mathbb{R})} .$$

1.2.1.2 Composition of a function with a distribution In the following, the construction of distributions with support on a light cone will be required. One way to achieve this is to adapt the contruction of Friedlander in [39] in the case of spinor valued distribution.

Definition 1.33. *Let S be a smooth function on Ω, with non vanishing gradient on Ω. Let f be a distribution with compact support on $\mathbb{R}$.*
Then, the application

$$\phi \in \mathcal{D}(\Omega) \longrightarrow \left(f(t), \int_{S(p)=t} \phi(p) \nabla S(p) \lrcorner \mu(p) \right)$$

where $\nabla S(p) \lrcorner \mu(p)$ is the contraction of the measure on M with the gradient ∇S (or the Leray measure on the hypersurface $S(p) = t$), defines a real distribution denoted $f(S)$. This distribution coincides with the composition of functions when f is represented by a function.

We need to apply this definition to calculate the action of the Dirac operator to a distribution on $\mathbb{S}_{Dirac}$ of the form $f(S)U$:

Proposition 1.34. *Let f be an element of $\mathcal{E}'(\mathbb{R})$, S a smooth function chosen as in definition 1.33 and U a smooth spinor field on M.*
Then, in the sense of distributions,

$$\not{D}\left(f(S)U\right) = f'(S)\hat{\nabla}(S)\cdot U + f(S)\not{D}U,$$

where $\hat{\nabla}S$ is the raised gradient, i.e.

$$\hat{\nabla}(S) := \sum_i \left(\nabla_{e_i} u\right) e_i.$$

.

Proof. : Let $\Phi \in \mathcal{D}(\mathbb{S}_{Dirac})$ and (f_i) an orthonormal frame. Φ is chosen with support in domain Ω where $\nabla_{f_i} f_i$ are all zero. We calculate $(\not{D}\left(f(S)U\right), \Phi)_{\mathcal{D}'(\mathbb{S}_{\mathrm{Dirac}}),\mathcal{D}(\mathbb{S}_{\mathrm{Dirac}})}$ using the previous definitions and lemma 1.29:

$$\begin{aligned}
(\not{D}\left(f(S)U\right), \Phi)_{\mathcal{D}'(\mathbb{S}_{\mathrm{Dirac}}),\mathcal{D}(\mathbb{S}_{\mathrm{Dirac}})} &= (f(S)U, \not{D}\Phi)_{\mathcal{D}'(\mathbb{S}_{\mathrm{Dirac}}),\mathcal{D}(\mathbb{S}_{\mathrm{Dirac}})} \\
&= < f, \int_{S(p)=t} (U, \not{D}\Phi)\mu_{S_t} >_{\mathcal{E}'(\mathbb{R}),\mathcal{E}(\mathbb{R})} \\
&= < f, \int_{S(p)=t} (\not{D}U, \Phi)\mu_{S_t} >_{\mathcal{E}'(\mathbb{R}),\mathcal{E}(\mathbb{R})} \qquad (1.5) \\
&+ \ < f, \int_{S(p)=t} \nabla_{f_i}(U, f_i\cdot\Phi)\mu_{S_t} >_{\mathcal{E}'(\mathbb{R}),\mathcal{E}(\mathbb{R})} \qquad (1.6)
\end{aligned}$$

where μ_{S_t} is the Leray measure $\nabla S \lrcorner \mu$ on the hypersurface $S_t = \{S(p) = t\}$. We calculate the two terms independently; by definition, (1.5) is:

$$< f, \int_{S(p)=t} (\not{D}U, \Phi)\mu_{S_t} >_{\mathcal{E}'(\mathbb{R}),\mathcal{E}(\mathbb{R})} = (f(S)\not{D}U, \Phi)_{\mathcal{D}'(\mathbb{S}_{Dirac}),\mathcal{D}(\mathbb{S}_{Dirac})}.$$

and (1.6) is calculated using the same idea as in lemma (1.29):

$$< f, \int_{S(p)=t} \nabla_{f_i}(U, f_i\cdot\Phi)\mu_{S_t} >_{\mathcal{E}'(\mathbb{R}),\mathcal{E}(\mathbb{R})} = < f, \int_{S(p)=t} \mathrm{div}(v)\mu_{S_t} >_{\mathcal{E}'(\mathbb{R}),\mathcal{E}(\mathbb{R})}$$

where v is the vector field on Ω defined by:

$$v = \sum_{i=0}^{3} \mathfrak{f}_i(U, f_i\cdot\Phi) f_i$$

with $\mathfrak{f}_i = < f_i, f_i >$. Noticing that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(p)\le t} \mathrm{div}(v)\mu = \int_{S_t} \mathrm{div}(v)\mu_{S_t}$$

and using the Stokes theorem

$$\begin{aligned}
\int_{S(p)\leq t} \operatorname{div}(v)\mu &= \int_{S_t} < \nabla S(p), v > \mu_{S_t} \\
&= \int_{S_t} \sum_i \nabla_{f_i} S(U, f_i \cdot \Phi)\mu_{S_t} \\
&= \int_{S_t} (U, \hat{\nabla} S \cdot \Phi)\mu_{S_t} \\
&= -\int_{S_t} (\hat{\nabla} S \cdot U, \Phi)\mu_{S_t},
\end{aligned}$$

we obtain, accordingly with definition 1.31:

$$< f, \int_{S_t} (\nabla_{f_i}(U, f_i \cdot \Phi)\mu_{S_t} >_{\mathcal{E}'(\mathbb{R}),\mathcal{E}(\mathbb{R})} = (f'(S)\hat{\nabla} S \cdot U, \Phi)_{\mathcal{D}'(\mathbb{S}_{\mathrm{Dirac}}),\mathcal{D}(\mathbb{S}_{\mathrm{Dirac}})}$$

so that:

$$\not{D}\,(f(S)U) = f'(S)\hat{\nabla}(S) \cdot U + f(S)\not{D}U \ \blacklozenge$$

1.2.1.3 Spinors and bidistributions Keeping in sight that the purpose is to write an integral formula (or representation formula) for a Cauchy problem, we must be able to apply twice a distribution to spinor fields. This is what bidistributions are made for.

We define the product $\boxtimes$ of two smooth sections of $\mathbb{S}_{Dirac}$ with compact support by:

$$\begin{array}{ccc}
\mathcal{D}(\mathbb{S}_{Dirac}) \times \mathcal{D}(\mathbb{S}_{Dirac}) & \longrightarrow & \mathcal{D}(\mathbb{S}_{Dirac}) \boxtimes \mathcal{D}(\mathbb{S}_{Dirac}) \\
(\Phi, \Psi) & \longmapsto & ((p,q) \in \Omega \times \Omega \mapsto \Psi(p) \otimes \Phi(q))
\end{array}$$

where $\otimes$ must be understood as tensor product of spinors in different variables. The vector space generated by these products is denoted by $\mathcal{D}(\mathbb{S}_{Dirac}) \boxtimes \mathcal{D}(\mathbb{S}_{Dirac})$.

Definition 1.35. *Let u and v be two distributions in $\mathcal{D}'(\mathbb{S}_{Dirac})$. The bidistribution $u \boxtimes v$ is an application from $\mathcal{D}(\mathbb{S}_{Dirac}) \boxtimes \mathcal{D}(\mathbb{S}_{Dirac})$ defined by, for every $(\phi, \psi) \in \mathcal{D}(\mathbb{S}_{Dirac}) \times \mathcal{D}(\mathbb{S}_{Dirac})$:*

$$(u \boxtimes v, \phi \boxtimes \psi) = (u, \phi)_{\mathcal{D}(\mathbb{S}_{Dirac}),\mathcal{D}'(\mathbb{S}_{Dirac})}(v, \psi)_{\mathcal{D}'(\mathbb{S}_{Dirac}),\mathcal{D}(\mathbb{S}_{Dirac})}.$$

The vector space $\mathcal{D}'(\mathbb{S}_{Dirac}) \boxtimes \mathcal{D}'(\mathbb{S}_{Dirac})$ generated by these products is called the space of spinor-valued bidistributions on $\mathbb{S}_{Dirac}$.

If ϕ is in $\mathcal{D}(\mathbb{S}_{Dirac})$ and u is a spinor bidistribution, then $u(\phi)$ is still in $\mathcal{D}'(\mathbb{S}_{Dirac})$. It can consequently be still applied to a function in $\mathcal{D}(\mathbb{S}_{Dirac})$.

A special type of spinor valued distribution that will be encountered in the following is the Dirac distribution.

Definition 1.36. *We define the Dirac distribution (or Dirac mass) in p', denoted by $\overline{\delta}_{p'}$ by:*

$$\forall \phi \in \mathcal{D}(\mathbb{S}), (\overline{\delta}_{p'}, \phi)_{\mathcal{D}'(\mathbb{S}_{Dirac}),\mathcal{D}(\mathbb{S}_{Dirac})} = \phi(p').$$

It must be noted that this distribution can be written in the form $\tau(p',p)\delta_{p'}$ ([39], chapter 6) where τ is a linear transformation from $\mathcal{D}(\mathbb{S}_{Dirac})$ in the variable p to $\mathcal{D}(\mathbb{S}_{Dirac})$ in the variable q satisfying $\tau(p',p') = I_{\mathbb{S}_{Dirac}}$ and can consequently be written as:

$$\forall\phi \in \mathcal{D}(\mathbb{S}), (\tau(p',p)\delta_{p'},\phi)_{p,\{\mathcal{D}'(\mathbb{S}_{Dirac}),\mathcal{D}(\mathbb{S}_{Dirac})\}} = \phi(p'),$$

the duality bracket being computed in the variable p. Since

$$\varepsilon^{AB}\varepsilon_A^{\ 0}\varepsilon_B^{\ 1} = 1 \text{ and } \varepsilon_{A'B'}\varepsilon_{0'}^{\ A'}\varepsilon_{1'}^{\ B'} = -1$$

and $\tau(p,p)$ satisfies:

$$(\tau(p,p),\phi(p)) = \phi(p)$$

it can be explicitly calculated at $p = p'$:

$$\begin{aligned} \tau(p,p) &= -\varepsilon_{1'}^{\ A'} \boxtimes \varepsilon_{0'}^{\ A'} + \varepsilon_{0'}^{\ A'} \boxtimes \varepsilon_{1'}^{\ A'} + \varepsilon_A^{\ 0} \boxtimes \varepsilon_A^{\ 1} - \varepsilon_A^{\ 1} \boxtimes \varepsilon_A^{\ 0} \\ &= -\bar{\iota}^{B'} \boxtimes \bar{o}^{A'} + \bar{o}^{B'} \boxtimes \bar{\iota}^{A'} - o_B \boxtimes \iota_A + \iota_B \boxtimes o_A. \end{aligned} \tag{1.7}$$

Such a function τ is chosen explicitly later (see equation (1.9)).

1.2.2 Fundamental solutions of the wave equation

We now apply to the spinorial wave equation the analytical tools used by Friedlander in [39] for the tensor wave equation. An alternative method has been used by Klainerman and Rodnianski to construct an approximate fundamental solution in [60]. Though their method is more flexible and well-suited to obtain estimates, it is not appropriate here since, as we will see, the regular part (the tail of the fundamental solution) is needed to write down a fundamental solution. V. Moncrief used Friedlander's method in a paper with D. Eardley ([33]) for the Yang-Mills equations in the Minkowski space and for the Maxwell wave equation in [65] on a curved space-time.

We first consider the spinorial wave operator $\not{D}^2$. The Schrödinger - Lichnerowicz - Böchner formula gives that for any ϕ in $\mathcal{D}(\mathbb{S}_{Dirac})$:

$$\not{D}^2\phi = \Box\phi + \frac{1}{4}\text{Scal}\phi \tag{1.8}$$

where $\Box = -\nabla_j\nabla^j$. Since the index notations are used from the beginning, a index version of the formula with its proof is given:

Proposition 1.37 (Schrödinger-Lichnerowicz formula in index version for spin $\frac{1}{2}$). *Let ϕ_A be a smooth section of $\mathbb{S}_A$. Then we have the following relation:*

$$\nabla_{BA'}\nabla^{AA'}\phi_A = \frac{1}{2}\left(\nabla_{CC'}\nabla^{CC'}\phi_B + \frac{1}{4}Scal_g\phi_B\right).$$

Proof. : The reader should refer for intermediate results to [79](4.9.2 and 4.9.17).

$$\begin{aligned} \nabla_{BA'}\nabla^{AA'}\phi_A &= \varepsilon^{AC}\nabla_{BA'}\nabla_C^{A'}\phi_A \\ &= \varepsilon^{AC}\left(\nabla_{[B|A'}\nabla_{|C]}^{A'}\phi_A + \nabla_{(B|A'}\nabla_{|C)}^{A'}\phi_A\right) \\ &= \frac{1}{2}\varepsilon^{AC}\nabla_{HH'}\nabla^{HH'}\varepsilon_{BC}\phi_A + \frac{1}{8}\text{Scal}_g\phi_B \text{ (formula 4.9.17 in [79])} \\ &= \frac{1}{2}\nabla_{CC'}\nabla^{CC'}\phi_B + \frac{1}{8}\text{Scal}_g\phi_B \blacklozenge \end{aligned}$$

Remark 1.38.
- *This version agrees with the previous one when noticing that the operator $\nabla_{BA'}\nabla^{AA'}$ is in fact, due to the renormalization induced by the Clifford multiplication, the projection on $\mathbb{S}_B$ of $1/2\not{D}^2$.*
- *A generalization of this formula to arbitrary spin is given later in subsection 3.1.*

Since Ω is a geodesically convex domain, it is possible to define globally on Ω the squared-distance function:

$$\Gamma_p(q) = \int_0^t g\left(\frac{\mathrm{d}\gamma(s)}{\mathrm{d}s}, \frac{\mathrm{d}\gamma(s)}{\mathrm{d}s}\right)\mathrm{d}s$$

where $\gamma : [0,t] \to \Omega$ is the unique geodesic from p to q.

To write the fundamental solutions of the wave equation, it is necessary to construct distributions with support on a cone: using definition 1.33, let us consider the distributions

$$\delta^{\pm}(\Gamma_p(q)) \text{ and } H^{\pm}(\Gamma_p(q))$$

where δ is the Dirac mass and H the Heaviside function. These distributions have support respectively, for p fixed in Ω, in $C^{\pm}(p)$ and $\mathcal{J}^{\pm}(p)$.

Remark 1.39. *It is important to notice that these distributions do not satisfy definition* 1.33 *since the gradient of* $\Gamma_p(q)$ *vanishes at the vertex of the cone. Nonetheless, considering the distributions*

$$\delta^{\pm}(\Gamma_p(q) - \varepsilon) \text{ and } H^{\pm}(\Gamma_p(q) - \varepsilon)$$

with ε positive avoids the problem. The results can then be obtained using a limiting process when ε tends to zero. This method will be used later to expand equation (2.8)*.*

It is known that the operator $\not{D}^2$ admits fundamental solutions ([39],[24]):

Theorem 1.40. *There exists two bidistributions on Ω, $\tilde{G}_q^{\pm}(p)$ that satisfy:*

$$\forall (p,q) \in \Omega^2, \not{D}_p^2 \tilde{G}_q^{\pm}(p) = \overline{\delta}_q(p)$$

in the distribution sense. These two bidistributions can be written:

$$\tilde{G}_q^{\pm}(p) = \tilde{U}_q(p)\delta^{\pm}(\Gamma_q(p)) + \tilde{V}_q(p)H^{\pm}(\Gamma_q(p)).$$

where $\tilde{U}$ and $\tilde{V}^{\pm}$ are smooth functions of the variable (p,q). q being fixed in Ω, the support of $\tilde{G}_q^{\pm}(p)$ is then in $\mathcal{C}^{\pm}(q)$.

The structure of the fundamental solution obtained by Friedlander is the following (the reader should refer to [39] for more details.)

1. The function $\tilde{U}$ in the singular part can be decomposed into two parts, $\tilde{U}_q(p) = k_q(p)\tilde{\tau}_q(p)$ where:

   (a) the bispinor $\tilde{\tau}_q(p)$ satisfies:

   $$\nabla^i\Gamma_q(p)\nabla_i\tilde{\tau}_q(p) = 0 \text{ and } \tilde{\tau}_p(p) = \tau_p(p). \tag{1.9}$$

   This equation can easily be reinterpreted as parallel transport in the variable q of the bispinor identity along the geodesic from p to q.

(b) the function $k_q(p)$ satisfies the transport equation:

$$2 < \nabla\Gamma_q(p), \nabla k_q(p) > +(\Box\Gamma_q(p) - 8)k_q(p) = 0 \text{ and } k_p(p) = \frac{1}{2\pi}. \qquad (1.10)$$

$k_q(p)$ measures the difference between the measure induced on $\mathcal{C}^+(p) \cap \mathcal{C}^-(q)$ and the measure on the standard sphere S^2 in the sense that, if p is in the future of q:

$$\mu_{\mathcal{C}^+(q)\cap\mathcal{C}^-(p)} = k_q(p) r^2 \mu_{S^2}$$

where $\mu_{\mathcal{C}^+(q)\cap\mathcal{C}^-(qp)}$ is the Riemannian volume form induced by the metric g on $\mathcal{C}^+(q) \cap \mathcal{C}^-(p)$ and μ_{S^2} the standard volume form on the two dimensional sphere.

2. The regular part $\tilde{V}^\pm$ of the fundamental solution can be obtained by solving the characteristic Cauchy problem:

$$\begin{cases} \Box\tilde{V}_q(p) & = & 0 \text{ for } p \in \mathcal{J}^+(q) \\ \tilde{V}_q(p) & = & \tilde{V}_q^0(p) \text{ for } p \in \mathcal{C}^+(q) \end{cases}$$

where $\tilde{V}_q^0(p)$ satisfies the transport equation:

$$2 < \nabla\Gamma_q(p), \nabla\tilde{V}_q^0(p) > +(\Box\Gamma_q(p) - 4)\tilde{V}_q^0(p) = -D^2\tilde{U}.$$

For later convenience, the fundamental solution must be split over the decomposition of the Dirac spinors:

$$\tilde{G}_q^\pm(p) = {}^1\tilde{\mathrm{G}}^{\pm}_{\underset{A}{q}\ B}(p) + {}^2\tilde{\mathrm{G}}^{\pm}{}_{\underset{q}{A'}}{}^{B'}(p)$$

The notation $\overset{q}{A}$ means that the part of the bidistribution in the variable q acts on spinor fields in $\mathbb{S}_A$. Their fundamental part is denoted by, respectively, ${}^1\tilde{\mathrm{U}}^{\pm}_{\underset{A}{q}\ B}(p)$ and ${}^2\tilde{\mathrm{U}}^{\pm}{}_{\underset{q}{A'}}{}^{B'}(p)$.

Two backward and forward fundamental solutions for the wave equation can then be constructed. For Dirac spinors, these fundamental solutions are the distributions:

$$D^p\tilde{G}_q^\pm(p)$$

In terms of indices, these fundamental solutions are written:

$$\nabla_p^{BB'}{}^1\tilde{\mathrm{G}}^{\pm}_{\underset{A}{q}\ B}(p) \text{ on } \mathbb{S}_A \boxtimes \mathbb{S}^{B'} \text{ and } \nabla^p_{BB'}{}^2\tilde{\mathrm{G}}^{\pm}{}_{\underset{q}{A'}}{}^{B'}(p) \text{ on } \mathbb{S}^{A'} \boxtimes \mathbb{S}_B.$$

Finally, we state the following theorem concerning the existence and the structure of the fundamental solution for the Dirac equation for Dirac spinors.

Theorem 1.41. *There exist two fundamental solutions for the Dirac operator $\not{D}$, $G_q^\pm(p)$, with support in $\mathcal{C}^\pm(q)$, for q fixed in Ω, such that:*

$$\forall(p,q) \in \Omega^2, \not{D}^p G_q^\pm(p) = \overline{\delta}_q(p)$$

in the distribution sense. These two fundamental solutions are obtained by applying the Dirac operator to the two fundamental solutions of the wave equation:

$$G_q^\pm(p) = \not{D}^p\tilde{G}_q^\pm(p).$$

2 Derivation of the integral formula for Dirac spinors

This section is devoted to the derivation of an integral formula for Dirac spinors for the characteristic Cauchy problem with data on a future null cone. In this context, we will work with the forward fundamental solution $G_q^+(p)$ which will be denoted with no ambiguity $G_q(p)$. The singular and smooth parts of the forward fundamental solution for the wave equation will be denoted $\tilde{U}_q(p)$ and $\tilde{V}_q(p)$.

The point p_0 being fixed, let p be a point in the future of p_0 in Ω. We define, for these two points:

- $\sigma(p) = \mathcal{C}^+(p_0) \cap \mathcal{C}^-(p)$
- $\mathcal{D}(p) = \mathcal{C}^+(p_0) \cap \mathcal{J}^-(p)$
- $\mathcal{S}(p) = \mathcal{J}^+(p_0) \cap \mathcal{C}^-(p)$.
- $\mathcal{V}(p) = \mathcal{J}^+(p_0) \cap \mathcal{J}^-(p)$

Since Ω is geodesically convex, these instersections are well-defined (in fact, the hypothesis of global hyperbolicity suffices).

2.1 Representation formula

The first step to obtain a representation formula is to solve the problem with source:

$$\not{D}u = f.$$

The following lemma is a direct transcription of lemma 5.5.1 in [39]:

Lemma 2.1. *Let f in $\mathcal{E}(\mathbb{S}_{Dirac})$.*
Then the distributions defined by:

$$\forall \phi \in \mathcal{D}(\mathbb{S}_{Dirac}), (u, \phi)_p := (f, (G_p^{\pm}, \phi)_q)_p$$

are solutions of the problem:

$$\not{D}u = f.$$

Proof. : The calculation is made first formally. The justication of each step will be carried out later; it will be sufficient to check that each duality bracket is well-defined and that all the operations involved (symmetry on Dirac operator, ...) are legitimate.
Let ϕ be in $\mathcal{D}(\mathbb{S}_{Dirac})$.

$$\begin{aligned} (\not{D}^p u, \phi)_p &= (u, \not{D}^p \phi)_p & (2.1)\\ &= (f, (G_p^{\pm}, \not{D}^q \phi)_q)_p \text{ by definition of } u & (2.2)\\ &= (f, \phi) \text{ by definition of } G_p^{\pm}. & (2.3) \end{aligned}$$

It must be checked to insure that (2.2) exists that the function:

$$p \longmapsto (G_p^{\pm}, \not{D}^q \phi)_q$$

is smooth; we have:

$$(\not{D}^q \tilde{G}_p^{\pm}, \not{D}^q \phi)_q = (\tilde{G}_p^{\pm}, (\not{D}^q)^2 \phi)_q = \int_{\mathcal{C}^+(p)} (\tilde{U}_p^{\pm}(q), (\not{D}^q)^2 \phi) \mu_{\Gamma_p(q)}(q) + \int_{\mathcal{J}^+(p)} (\tilde{V}_p^{\pm}(q), (\not{D}^q)^2 \phi) \mu(q), \tag{2.4}$$

where $\mu_{\Gamma_p(q)}$ is the Leray form associated with the function $\Gamma_p(q)$, i.e:

$$\mu_{\Gamma_p(q)} = \nabla^q \Gamma_p(q) \lrcorner \mu.$$

Let $\pi : \Omega \to \mathbb{R}^4$ be a chart recovering Ω (which exists since Ω is geodesically convex). The image by π of p and q are respectively denoted by y and x. There exists a diffeomophism $\xi \to x = h(y, \xi)$ from $\pi(\Omega)$ into $\mathbb{R}^4$, where $\xi = (\xi^0, \xi^1, \xi^2, \xi^3)$ is a coordinate system centered at y, Minkowskian in q and such that the vector $(1, 0, 0, 0)$ is timelike and future oriented. In this coordinate system, the measures μ and $\mu_{\Gamma_q(p)}$ are expressed as:

$$\mu(q) = k(y, \xi)\mathrm{d}\xi \text{ and } \mu_{\Gamma_q(p)} = k(y, \xi)\frac{\mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2 \wedge \mathrm{d}\xi^3}{2\sqrt{(\xi^1)^2 + (\xi^2)^2 + (\xi^3)}}$$

with $\mathrm{d}\xi = \mathrm{d}\xi^0 \wedge \mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2 \wedge \mathrm{d}\xi^3$ and

$C^+(q) = \{\xi | \xi^0 = \sqrt{(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2}\}$ and $\mathcal{J}^+(q) = \{\xi | \xi^0 \geq \sqrt{(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2}\}$.

The integral (2.4) can then be rewritten:

$$(\not{D}^q \tilde{G}_p^\pm, \not{D}^q \phi)_q = \int_{\xi^0 = \sqrt{(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2}} (\tilde{U}^\pm_{h(y,\xi)}(y), \big((\not{D}^q)^2 \phi\big)(h(y, \xi)))k(y, \xi)\frac{\mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2 \wedge \mathrm{d}\xi^3}{2\sqrt{(\xi^1)^2 + (\xi^2)^2 + (\xi^3)}} \tag{2.5}$$

$$+ \int_{\xi^0 \geq \sqrt{(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2}} (\tilde{V}^\pm_{h(y,\xi)}(y), \big((\not{D}^q)^2 \phi\big)(h(y, \xi))k(y, \xi)\mathrm{d}\xi. \tag{2.6}$$

which is clearly a smooth function of $y = \pi(p)$.
Since f is a distribution with compact support, there exists K', an integer N and a positive constant C such that, for any smooth function ψ, the following estimate holds:

$$|(f, \psi)| \leq C \sum_{|\alpha| < N} \sup_{y \in \pi(K')} ||\partial_y^\alpha \psi \circ \pi^{-1}(y)||.$$

Let K be a compact of Ω. Assume that ϕ has its support in K. Then the previous inequality gives for $\psi = (G_p, \not{D}\phi)_q$

$$|(f, \psi)| \leq C' \sum_{|\alpha| < N} \sup_{y \in \pi(K')} ||\partial_y^\alpha (G_{\pi^{-1}(y)}(\pi^{-1}(x)), \phi)_x||.$$

Using the expression of $(G_{\pi^{-1}(y)}(\pi^{-1}(x)), \phi)_x$, its derivatives $\partial_y^\alpha (G_{\pi^{-1}(y)}(\pi^{-1}(x)), \phi)_x$ are bounded by the derivatives of ϕ on K:

$$\sup_K ||\partial_y^\alpha (G_{\pi^{-1}(y)}(\pi^{-1}(x)), \phi)_x|| \leq C_{K,K',\alpha} \sum_{|\beta \leq |\alpha| + 1} \sup_{y \in \pi(K)} ||\partial_y^\beta \phi \circ \pi^{-1}(y)||$$

where the constant $C_{K,K',\alpha}$ is determined only by the derivatives of $\tilde{U}$, $\tilde{V}$, h of order up to $k + 1$ on the compact $K \times K'$ and its image by π. Finally, we obtain:

$$|(f, \psi)| \leq \sum_{|\alpha| < N+1} \sup_{y \in \pi(K')} ||\partial_y^\alpha (\phi \circ \pi^{-1})||,$$

which means that equation (2.2) is well-defined.⧫

Let u be in $\mathcal{E}(\mathbb{S}_{Dirac})$. The following proposition gives a representation of u in term of its data on a null cone:

Theorem 2.2. *Let u be a function with future bounded support. Let p_0 in Ω. Then, we have in the distributional sense:*

$$uH^+(\Gamma_0) = \not{D}^q \left(\left(\left(\int_{\mathcal{S}(q)} (\not{D}^p u, \tilde{U}_p(q))\mu_{\Gamma_q}(p) + \int_{\mathcal{V}(q)} (\not{D}^p u, \tilde{V}_p(q))\mu_{\Gamma_q}(p) \right.\right.\right.$$
$$\left.\left.\left. + \int_{\sigma(q)} (\nabla^p\Gamma_0 \cdot u, \tilde{U}_p(q))\mu_{\Gamma_0,\Gamma_q}(p) + \int_{\mathcal{D}(q)} (\nabla^p\Gamma_0 \cdot u, \tilde{V}_p(q))\mu_{\Gamma_0}(p) \right) H^+(\Gamma_0) \right) \right),$$

where the two-form $\mu_{\Gamma_0,\Gamma_q}(p)$ is obtained via the factorization:

$$\forall \phi \in C_0^\infty(\Omega \times \Omega), \int_A \phi\mu_{\Gamma_0}(p) \wedge \mu_\Gamma(q) = \int_{\mathcal{J}^+(p_0)} \int_{\sigma(q)} \phi\mu_{\Gamma_0,\Gamma_q}(p) \wedge \mu(q),$$

where A is the set $\{(p,q)|p \in \mathcal{C}^+(p_0)$ and $q \in \mathcal{C}^+(p)\}$.

Proof. Let u be a function with future bounded support, that is to say that the intersection of $supp(u)$ with any future null cone is compact . We use here the property of the fundamental solution with lemma 2.1 with $f = \not{D}^p(uH^\pm(\Gamma_0))$:

$$\begin{aligned}(uH^+(\Gamma_0), \phi)_p &= \left(\not{D}^p\left(uH^+(\Gamma_0)\right), (G_p, \phi)_q\right)_p &(2.7)\\ &= \left(\not{D}^p\left(uH^+(\Gamma_0)\right), \left(\not{D}^q\tilde{G}_p, \phi\right)_q\right)_p \\ &= \left(\not{D}^p\left(uH^+(\Gamma_0)\right), \left(\tilde{G}_p, \not{D}^q\phi\right)_q\right)_p &(2.8)\end{aligned}$$

The duality bracket (2.8) is properly defined since the function $p \mapsto (\tilde{G}_p, \not{D}^q\phi)_q$ is a smooth function with support in the future of Supp(ϕ), that is to say $\cup_{q\in \mathrm{Supp}(\phi)}\mathcal{I}^+(q)$, and since u has future bounded support.

The duality bracket (2.8) is then developed. The first step consists in differentiating the distributions $uH^+(\Gamma_0)$. As already noticed in remark 1.39, the distribution $uH^+(\Gamma_0)$ is not of the type given in proposition 1.34 since $\nabla\Gamma_0$ vanishes at p_0. To avoid this difficulty, we consider the distributions $uH^+(\Gamma_0 - \varepsilon)$, where ε is a positive number. This derivation gives, since proposition 1.34 can be applied:

$$\not{D}\left(uH^+(\Gamma_0 - \varepsilon)\right) = (\not{D}u)H^+(\Gamma_0 - \varepsilon) + \hat{\nabla}\Gamma_0 \cdot u\delta^+(\Gamma_0 - \varepsilon)$$

which becomes, when ε tends to zero:

$$\not{D}\left(uH^+(\Gamma_0)\right) = (\not{D}u)H^+(\Gamma_0) + \hat{\nabla}\Gamma_0 \cdot u\delta^+(\Gamma_0).$$

The bracket (2.8) is written as the sum of four integrals:

$$\begin{aligned}\left(\not{D}^p\left(uH^+(\Gamma_0)\right), \left(\tilde{G}_p, \not{D}^q\phi\right)_q\right)_p &= \int_{\mathcal{J}^+(p_0)} \int_{\mathcal{C}^+(p)} (\not{D}^p u, (\tilde{U}_p(q), \not{D}^q\phi))\mu_{\Gamma_p}(q) \wedge \mu(p)\\ &+ \int_{\mathcal{J}^+(p_0)} \int_{\mathcal{J}^+(p)} (\not{D}^p u, (\tilde{V}_p(q), \not{D}^q\phi))\mu(q) \wedge \mu(p)\\ &+ \int_{\mathcal{C}^+(p_0)} \int_{\mathcal{C}^+(p)} (\nabla^p\Gamma_0 \cdot u, (\tilde{U}_p(q), \not{D}^q\phi))\mu_{\Gamma_p}(q) \wedge \mu_{\Gamma_0}(p)\\ &+ \int_{\mathcal{C}^+(p_0)} \int_{\mathcal{J}^+(p)} (\nabla^p\Gamma_0 \cdot u, (\tilde{V}_p(q), \not{D}^q\phi))\mu(q) \wedge \mu_{\Gamma_0}(p),\end{aligned}$$

where μ_{Γ_0} and μ_{Γ_p} are the Leray measures associated with Γ_0 and Γ_p respectively.

Switching the order of integration of the variables, we get:

$$\begin{aligned}\left(\not{D}^p\left(uH^+(\Gamma_0)\right), \left(\tilde{G}_p, \not{D}^q\phi\right)_q\right)_p &= \int_{\mathcal{J}^+(p_0)}\int_{\mathcal{C}^-(q)\cap\mathcal{J}^+(p_0)} \left((\not{D}^p u, \tilde{U}_p(q)), \not{D}^q\phi\right)\mu_{\Gamma_q}(p)\wedge\mu(q)\\ &+ \int_{\mathcal{J}^+(p_0)}\int_{\mathcal{J}^-(q)\cap\mathcal{J}^+(p_0)} \left((\not{D}^p u, \tilde{V}_p(q)), \not{D}^q\phi\right)\mu(p)\wedge\mu(q)\\ &+ \int_{\mathcal{J}^+(p_0)}\int_{\mathcal{C}^-(q)\cap\mathcal{C}^+(p_0)} \left(\nabla^p\Gamma_0\cdot u, \tilde{U}_p(q)), \not{D}^q\phi\right)\mu_{\Gamma_0,\Gamma_q}(p)\wedge\mu(q)\\ &+ \int_{\mathcal{J}^+(p_0)}\int_{\mathcal{J}^-(q)\cap\mathcal{C}^+(p_0)} \left((\nabla^p\Gamma_0\cdot u, \tilde{V}_p(q)), \not{D}^q\phi\right)\mu_{\Gamma_0}(p)\wedge\mu(q).\end{aligned}$$

Finally, the duality bracket (2.7) is:

$$\begin{aligned}(uH^+(\Gamma_0),\phi) = \Bigg(\Bigg(\Bigg(&\int_{\mathcal{S}(q)}(\not{D}^p u, \tilde{U}_p(q))\mu_{\Gamma_q}(p) + \int_{\mathcal{V}(q)}(\not{D}^p u, \tilde{V}_p(q))\mu_{\Gamma_q}(p)\\ &+\int_{\sigma(q)}(\nabla^p\Gamma_0\cdot u, \tilde{U}_p(q))\mu_{\Gamma_0,\Gamma_q}(p) + \int_{\mathcal{D}(q)}(\nabla^p\Gamma_0\cdot u, \tilde{V}_p(q))\mu_{\Gamma_0}(p)\Bigg)H^+(\Gamma_0), \not{D}^q\phi\Bigg)_q\end{aligned}$$

which means that, in the sense of distributions, u satisfies, using the symmetry of the operator $\not{D}^q$:

$$\begin{aligned}uH^+(\Gamma_0) = \not{D}^q\Bigg(\Bigg(&\int_{\mathcal{S}(q)}(\not{D}^p u, \tilde{U}_p(q))\mu_{\Gamma_q}(p) + \int_{\mathcal{V}(q)}(\not{D}^p u, \tilde{V}_p(q))\mu_{\Gamma_q}(p)\\ &+\int_{\sigma(q)}(\nabla^p\Gamma_0\cdot u, \tilde{U}_p(q))\mu_{\Gamma_0,\Gamma_q}(p) + \int_{\mathcal{D}(q)}(\nabla^p\Gamma_0\cdot u, \tilde{V}_p(q))\mu_{\Gamma_0}(p)\Bigg)H^+(\Gamma_0)\Bigg).\blacklozenge\end{aligned}$$

A direct application of the previous theorem is the first integral formula for the characterictic Cauchy problem:

Proposition 2.3. *Let u be a smooth solution of:*

$$\not{D}u = 0$$

Then u can be expressed in $\mathcal{J}^+(p_0)$ in function of its restriction to the cone $\mathcal{C}^+(p_0)$ by:

$$u|_{\mathcal{J}^+(p_0)} = \not{D}^q\left(\left(\int_{\sigma(q)}(\nabla^p\Gamma_0\cdot u, \tilde{U}_p(q))\mu_{\Gamma_0,\Gamma_q}(p) + \int_{\mathcal{D}(q)}(\nabla^p\Gamma_0\cdot u, \tilde{V}_p(q))\mu_{\Gamma_0}(p)\right)H^+(\Gamma_0)\right).$$

Remark 2.4.

1. *This formula is not the final stage of our calculation; the fact that it only depends on initial conditions will be stated later. This is the purpose of the next subsection.*

2. *The vector $\nabla\Gamma_0$ being null along the cone $\mathcal{C}^+(p_0)$, Clifford multiplying with $\hat{\nabla}\Gamma_0$ means in fact contracting with the spinor form of $\nabla\Gamma_0$; a direct consequence of this is the fact the Clifford product of the 4-components Dirac spinors with $\nabla\Gamma_0$ does only involve the two components, u_0 and $u^{1'}$. The two remaining components are recovered using the constraints equations (cf. lemma 3.7 below).*

2.2 Integral formula

The integral formula is derived in three steps:

- construction of the appropriate geometric tools (derivation of measures, spin basis);
- interversion of the integral and the Dirac operator;
- and finally obtention of an expression of the singular part in terms of geometric quantities and initial data.

2.2.1 Geometric data on the cone

This section is devoted to the calculation of the relevent geometric quantities for the intersection of $\mathcal{C}^+(p_0) \cap \mathcal{C}^-(q) = \sigma(q)$ for a given point q in the future of p_0. This is widely inspired by section 4.14 of [79]. There are also some calculations of interest in the work of Frittelli, Newman and al ([42], for instance) and Nurowski – Robinson([75]). This kind of calculation is also very common in the study of Ricci flows.

We first choose a parallely transported vector field l along the null cone $\mathcal{C}^+(p_0)$:

$$\nabla_l l = 0.$$

We then consider, for a given point q in $\mathcal{J}^+(p_0)$, a point p in $\sigma(q)$. We construct at p a Newman-Penrose tetrad:

1. the first null vector is the vector $l(p)$ at p;
2. $n(p)$ is chosen on the future oriented null geodesic from p to q such that $g(l, n) = 1$;
3. we complete the basis by taking a pair of complex null vectors $m(p)$ and $\overline{m}(p)$ in the orthogonal of the vector space generated by (l, n) such that $g(m, \overline{m}) = -1$.

A Newman-Penrose tetrad is then obtained at each point q' on the cone $\mathcal{C}^-(q)$: let p' be the point in $\sigma(q)$ lying on the unique null geodesic from q' to q; the Newman-Penrose tetrad is obtained in q' by parallely transporting the one at p' along the unique null geodesic from p' to q'.

Remark 2.5. *This construction cannot be realized globally on the intersection $\mathcal{C}^+(p_0) \cap \mathcal{C}^-(q) = \sigma(q)$ which has the topology of $\mathbb{S}^2$. It will be necessary to make this construction on two different open sets and then glue these constructions together to obtain the result which only depends on l and n. We assume then that the construction is done on one open set.*

This choice of Newman Penrose tetrad gives us:

- a basis of $T\Omega \otimes \mathbb{C}$ and, consequently, up to a sign, a spin basis of $\mathbb{S}^A$ that will be denoted by (o^A, ι^A);
- if q is fixed first and p is chosen on $\sigma(q)$, the vectors m and $\overline{m}$ span the tangent plane to $\sigma(q)$ at p:
$$T_p\sigma(q) = \{\lambda\overline{m} + \overline{\lambda}m | \lambda \in \mathbb{C}\};$$
due to obvious topological obstructions (see remark 2.5), this construction cannot be extended globally to all $\sigma(q)$.

- the choice of l, which is parallely transported along the generators of $\mathcal{C}^+(p_0)$, and n, which is parallely transported along the generators of $\mathcal{C}^-(q)$, gives rise to two affine parameters r_0 and r along the null geodesics on these two cones.
- these two affine parameters give rise to two parametrizations by the sphere S^2 of $\sigma(q)$ using the exponential map at p_0 and p respectively:

$$\begin{array}{rccc} \exp_{p_0}: & S^2 & \longrightarrow & \Omega \\ & \omega & \longmapsto & \exp_{p_0}(r_0(\omega)\omega) \end{array}$$

and

$$\begin{array}{rccc} \exp_{p}: & S^2 & \longrightarrow & \Omega \\ & \omega & \longmapsto & \exp_{p}(r(\omega)\omega) \end{array}$$

Let q be a point fixed in $\mathcal{J}^+(p_0)$. We consider a point p on $\sigma(q)$. In a neighborhood of p, on $\sigma(q)$, is defined a Newman-Penrose tetrad $(l, n, m, \overline{m})$. The dual basis in $T\Omega^* \otimes \mathbb{C}$ is denoted by $(L, N, M, \overline{M})$ for which the following lemmata are true:

Lemma 2.6. *The induced metric on $\sigma(p)$ is $-2M\overline{M}$, the volume form $\frac{1}{2i}M \wedge \overline{M}$ and the mean curvature vector:*

$$H = 2(\rho' l + \rho n)$$

where ρ and ρ' are the real spin coefficients:

$$\rho = -(l, \nabla_{\overline{m}} m) \text{ and } \rho' = -(n, \nabla_m \overline{m})$$

Proof. : These results are straightforward consequences of the presentation concerning two-surfaces in [79] (section 4.14, proposition 4.14.2 sqq.)
The reality of the spin coefficients is stated in proposition (4.14.2) of [79], whenever l and n are orthogonal to a spacelike 2-surface (here $\sigma(q)$).
Since $(m, \overline{m})$ span $T\sigma(q)$, the second fundamental form is:

$$\forall (X, Y) \in T\sigma(q), II(X, Y) = (\nabla_X Y, n)l + (\nabla_X Y, l)n$$

so that the mean curvature vector is:

$$\begin{aligned} H &= -(II(m, \overline{m}) + II(\overline{m}, m)) \\ &= -\big(((\delta' m, n) + (\delta\overline{m}, n))l + ((\delta' m, l) + (\delta\overline{m}, l))n\big) \end{aligned}$$

Since (see [79] (4.5.28) together with (4.5.29)):

$$\begin{aligned} \delta' m &= (\beta - \overline{\alpha})m - \overline{\rho}' l - \rho n \\ \delta\overline{m} &= (\overline{\alpha} - \beta)m - \rho' l - \overline{\rho} n \end{aligned}$$

and since ρ and ρ' are real, we obtain:

$$H = 2(\rho' l + \rho n). \blacklozenge$$

In order to compute the Leray forms associated with the distance function, we use the expressions of the gradients of the distance functions Γ_0 and Γ_q:

$$\nabla^p \Gamma_0(p) = 2r_0 l(p) \text{ and } \nabla^p \Gamma_q(p) = 2rn(p) \tag{2.9}$$

The Leray forms can then be expressed using the dual basis of the chosen Newman-Penrose basis.

Proposition 2.7. *The Leray forms μ_{Γ_0}, μ_{Γ_q} and $\mu_{\Gamma_0,\Gamma}$ are:*

$$\begin{aligned}\mu_{\Gamma_0} &= \frac{1}{2ir_0} N \wedge M \wedge \overline{M} \\ \mu_{\Gamma_q} &= \frac{1}{2ir} L \wedge M \wedge \overline{M} \\ \mu_{\Gamma_0,\Gamma} &= \frac{1}{4ir_0 r} M \wedge \overline{M} = \frac{1}{4r_0 r}\mu_{\sigma(p)}\end{aligned}$$

where $\nabla^p \Gamma_0(p) = 2r_0 l(p)$ *and* $\nabla^p \Gamma_q(p) = 2rn(p)$

Proof. : The volume form on Ω can be expressed in terms of the Newman-Penrose tetrad as:

$$\mu = \frac{1}{i} L \wedge N \wedge M \wedge \overline{M}$$

so that, since $\mathrm{d}\Gamma_0 = 2r_0 L$ and $\mathrm{d}\Gamma_q = 2rN$, we obtain immediately:

$$\mu_{\Gamma_0} = \frac{1}{2ir_0} N \wedge M \wedge \overline{M} \text{ and } \mu_{\Gamma_q} = \frac{1}{2ir} L \wedge M \wedge \overline{M}.$$

The calculation of $\mu_{\Gamma_0,\Gamma}$ is obtained through the factorization given by Fubini's theorem:

$$\forall \phi \in C_0^\infty(\Omega \times \Omega), \int_A \phi \mu_{\Gamma_0}(p) \wedge \mu_\Gamma(q) = \int_{\mathcal{J}^+(p_0)} \int_{\sigma(q)} \phi \mu_{\Gamma_0,\Gamma_q}(p) \wedge \mu(q),$$

where A is the set $\{(p,q) | p \in \mathcal{C}^+(p_0) \text{ and } q \in \mathcal{C}^+(p)\}$. We get the desired expression of μ_{Γ_0,Γ_q}:

$$\mu_{\Gamma_0,\Gamma_q} = \frac{1}{4ir_0 r} M \wedge \overline{M} = \frac{1}{4r_0 r}\mu_{\sigma(q)}. \blacklozenge$$

The next step consists in determining the variation of the metric $\mu_{\sigma(q)}$ when q is in $\mathcal{I}^+(p_0)$. We first establish the technical lemma:

Lemma 2.8. *Let $(\mathcal{N}, h)$ be a smooth semi-riemanian manifold with metric h and Levi-Cevita connexion D; let X be a smooth vector field on $\mathcal{N}$. Let $(\mathcal{M}_p, g)$ be a submanifold of $\mathcal{N}$ such that the g metric induced by h is non degenerate and depending smoothly on a parameter p in $\mathcal{N}$ in the sense that there exists a smooth manifold Σ and a smooth map $f : \mathcal{N} \times \Sigma \longrightarrow \mathcal{N}$ which satisfies: $f(p, \star)$ is an immersion and $f(p, \Sigma) = \mathcal{M}_p$.*
We denote by μ_p the induced volume form on $\mathcal{M}_p$.
Then:

$$D_X^p \mu_p = -h(H, X)\mu_p$$

where H is the mean curvature vector field on $\mathcal{M}_p$.

Proof. : The Levi-Cevita connection induced by g on $\mathcal{M}_p$ is denoted ∇.
Let p be a point in $\mathcal{N}$ and q a point in $\mathcal{M}_p$. We choose around q a map $(V, (x^1, x^2, \dots, x^n), (x^{n+1}, \dots, x^{n+k}))$ normal at q and $\mathcal{M}_p \cap V = \{x^{n+1} = \cdots = x^{n+k} = 0\}$ such that, at q:

$$\nabla_{\partial_{x^i}} \partial_{x^i} = 0. \tag{2.10}$$

The volume form on $\mathcal{M}_p$ around q can be expressed:

$$\mu_p = |\det(g_{ij})|^{\frac{1}{2}} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n.$$

We calculate the derivative:

$$D_X\mu_p = \mathrm{Sign}(\det(g_{ij}))\frac{g^{ij}D_Xg_{ij}}{2\det(g_{ij})}|\det(g_{ij})|^{\frac{1}{2}}\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^n$$

and then evaluate at q, where the coordinate system is normal:

$$D_X\mu_p = \sum_{i=1}^{n}\frac{1}{2}\varepsilon_i D_X g(\partial_{x^i},\partial_{x^i})\mu_p$$

with $\varepsilon_i = g(\partial_{x^i},\partial_{x^i})$. Since the connection D on $\mathcal{N}$ is metric, we get:

$$\begin{aligned}
D_X\mu_p &= \sum_{i=1}^{n}\varepsilon_i h(D_X\partial_{x^i},\partial_{x^i})\mu_p\\
&= \sum_{i=1}^{n}\varepsilon_i h(D_{\partial_{x^i}}X+[X,\partial_{x^i}],\partial_{x^i})\mu_p\\
&= \sum_{i=1}^{n}\varepsilon_i(D_{\partial_{x^i}}h(X,\partial_{x^i})-h(X,D_{\partial_{x^i}}\partial_{x^i})+h([X,\partial_{x^i}],\partial_{x^i}))\mu_p
\end{aligned}$$

Since $D_{\partial_{x^i}}\partial_{x^i} = \nabla_{\partial_{x^i}}\partial_{x^i} + II(\partial_{x^i},\partial_{x^i})$, we finally obtain:

$$\nabla_X\mu_p = -h(X,H)\mu_p + \sum_{i=1}^{n}\varepsilon_i(D_{\partial_{x^i}}h(X,\partial_{x^i})-h(X,\nabla_{\partial_{x^i}}\partial_{x^i})+h([X,\partial_{x^i}],\partial_{x^i})\mu_p.$$

We then notice that, for i in $\{t1,n\}$:

$$h([X,\partial_{x^i}],\partial_{x^i}) = -\varepsilon_i\frac{\partial X^i}{\partial x^i} \text{ and } D_{\partial_{x^i}}h(X,\partial_{x^i}) = \varepsilon_i\frac{\partial X^i}{\partial x^i}.$$

Since $\nabla_{\partial_{x^i}}\partial_{x^i}$ is vanishing at q, the only remaining term is:

$$\nabla_X\mu_p = -h(X,H)\mu_p.\blacklozenge$$

A straightforward application of this lemma is the proposition:

Proposition 2.9. *Let $f:\Omega^2\to\mathbb{S}_{Dirac}$ be a smooth mapping.*
Then the following formula holds:

$$\not{D}^q\int_{\sigma(q)}f(q,p)\mu_{\sigma(q)}(p) = \int_{\sigma(q)}\not{D}^q f(q,p) + \nabla^p r_0\cdot\hat{\nabla}^p_l f(q,p) - 2\rho\hat{\nabla}^q r_0\cdot f(q,p)\mu_{\sigma(q)}(p)$$

where r_0, being a function of both p and q, satisfies $\nabla\Gamma_0 = 2r_0 l$.

Proof. : Let V a vector field on Ω. We work with the exponential map centered at p_0. $\sigma(q)$ can then be parametrized by S^2:

$$\omega\mapsto\exp_{p_0}(r_0(q,\omega)\omega).$$

Let us consider the variation of $\sigma(q)$ defined by, for some positive ε:

$$\begin{array}{lll}]-\varepsilon,\varepsilon[\times S^2 & \to & \Omega\\ (t,\omega) & \mapsto & \exp_{p_0}(r_0(q+tV,\omega)\omega) \end{array}.$$

Since

$$\begin{aligned}\nabla_V^q\big(f(q,p)\big) &= \nabla_V^q\big(f(q,\exp_{p_0}(r_0(q,\omega)\omega))\big)\\ &= \nabla_V^q f(q,p) + \left(\frac{\mathrm{d}}{\mathrm{d}t} f(q,\exp_{p_0}(r_0(q+tV,\omega)\omega)\right)\big|_{t=0}\end{aligned}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} f(q,\exp_{p_0}(r_0(q+tV,\omega)\omega)\right)\big|_{t=0} = \nabla_V^q r_0 \nabla_l^p f(q,p),$$

this gives, using lemma 2.6:

$$\nabla_V^q \int_{\sigma(q)} f(q,p)\mu_{\sigma(q)}(p) = \int_{\sigma(q)} \nabla_V^q f(q,p) + \nabla_V^q r_0 \nabla_l^p f(q,p) - 2\rho \nabla_V^q r_0 f(q,p)\mu_{\sigma(q)}$$

so that, when choosing an orthonormal basis (e_i) on Ω, we obtain:

$$\not{D}^q \int_{\sigma(q)} f(q,p)\mu_{\sigma(q)}(p) = \int_{\sigma(q)} \not{D}^q f(q,p) + \hat{\nabla}^q r_0 \cdot \nabla_l^p f(q,p) - 2\rho\hat{\nabla}^q r_0 \cdot f(q,p)\mu_{\sigma(q)}(p). \blacklozenge$$

We finally establish the following proposition:

Proposition 2.10. *Let $f:\Omega^2 \to \mathbb{S}_{Dirac}$ be a smooth mapping.*
Then the following formula holds:

$$\not{D}^q \int_{\mathcal{D}(q)} f(q,p)\mu_{\Gamma_0}(p) = \int_{\mathcal{D}(q)} \not{D}^q f(q,p)\mu_{\Gamma_0}(p) + \int_{\sigma(q)} \hat{\nabla}^q r_0 \cdot f(q,p)\frac{\mu_{\sigma(q)}(p)}{2r},$$

where r_0 and r, being functions of both p and q, satisfy $\nabla^p\Gamma_0 = 2r_0 l$ and $\nabla^p\Gamma_q = 2rn$.

Proof. : We use exactly the same method as in the proof of proposition 2.9. Using the parametrization of the exponential map centered at p_0, we have:

$$\begin{aligned}\not{D}^q \int_{\mathcal{D}(q)} f(q,p)\mu_{\Gamma_0}(p) &= \not{D}^q \int_{S^2}\int_0^{r_0(q,\omega)} f(q,r\omega)\frac{\mathrm{d}r}{2r}k(\omega,r)\mathrm{d}\omega_{S^2}\\ &= \int_{\mathcal{D}(q)} \not{D}^q f(q,p)\mu_{\Gamma_0}(p) + \int_{S^2} \hat{\nabla}^q r_0 \cdot f(q,r\omega)\frac{k(\omega,r)\mathrm{d}\omega_{S^2}}{2r}\\ &= \int_{\mathcal{D}(q)} \not{D}^q f(q,r\omega)\mu_{\Gamma_0}(p) + \int_{\sigma(q)} \hat{\nabla}^q r_0 \cdot f(q,p)\frac{\mu_{\sigma(q)}(p)}{2r}. \blacklozenge\end{aligned}$$

2.2.2 Derivation of the integral formula

We now consider the characteristic Cauchy problem on Ω:

$$\left\{\begin{array}{lcl}\not{D}u &=& 0 \text{ on } \mathcal{J}^+(p_0)\\ u &=& \theta \text{ on } \mathcal{C}^+(p_0)\end{array}\right. \tag{2.11}$$

where θ is a smooth spinor field on $\mathcal{C}^+(p_0)$, whose support does not encounter the vertex of the cone and satisfies the constraint equations given by lemma 3.7.

Remark 2.11. *The term "spinor field on the cone" must be understood as "trace on the cone" of a Dirac spinor field on Ω and not as a spinor field constructed as spinors on the manifold $\mathcal{C}^+(p_0)$.*

The basis constructed in the previous section is used to split the spinors:

$$\theta = \xi^{\mathbf{I}'}\varepsilon_{\mathbf{I}'}^{A'} + \zeta_{\mathbf{I}}\varepsilon_{A}^{\mathbf{I}} = \xi^{0'}\bar{o}^{A'} + \xi^{1'}\bar{\iota}^{A'} + \zeta_0(-\iota_A) + \zeta_1 o_A. \tag{2.12}$$

u will be split on $\mathbb{S}_A \otimes \mathbb{S}^{A'}$:

$$\begin{aligned} u &= \phi_A + \psi^{A'} \\ u &= \phi_0(-\iota_A) + \phi_1 o_A + \psi^{0'}\bar{o}^{A'} + \psi^{1'}\bar{\iota}^{A'}. \end{aligned}$$

The solution of (2.11) can be written in function of its data on the cone and the basis (o^A, ι^A):

Theorem 2.12. *Let u be a solution of (2.11). Then, for any q in $\mathcal{J}^+(p_0)$:*

$$\begin{aligned} u(q) = &\int_{\sigma(q)} \left(\frac{k_p(q)}{r}\right)\left(\nabla_l^p \xi^{1'}(p) - 2\rho\xi^{1'}(p)\right)\left(\nabla_q^{AA'} r_0\right) o_A(q)\mu_{\sigma(q)}(p) \\ &+ \int_{\sigma(q)} \xi^{1'}(p)\nabla^{AA'}\left(\left(\frac{k_p(q)}{r}\right) o_A(q)\right)\mu_{\sigma(q)}(p) \\ &+ \int_{\sigma(q)} \left(\frac{k_p(q)}{r}\right)\left(\nabla_l^p \zeta_0(p) - 2\rho\zeta_0(p)\right)\left(\nabla_{AA'}^q r_0\right)\bar{o}^{A'}(q)\mu_{\sigma(q)}(p) \\ &+ \int_{\sigma(q)} \zeta_0(p)\nabla_{AA'}\left(\left(\frac{k_p(q)}{r}\right)\bar{o}^{A'}(q)\right)\mu_{\sigma(q)}(p) + \int_{\sigma(q)} \hat{\nabla}^q r_0 \cdot \left(\nabla^p\Gamma_0 \cdot u, \tilde{V}_p(q)\right)\frac{\mu_{\sigma(q)}(p)}{2r} \\ &+ \int_{\mathcal{D}(q)} (\not{D}^p \tilde{V}_q, \hat{\nabla}^p\Gamma_0 \cdot u)\mu_{\Gamma_0}(p) \end{aligned}$$

Remark 2.13. *It is possible to obtain a representation formula for the Goursat problem for the Weyl equation:*

$$\nabla^{AA'}\phi_A = 0$$

by projecting the solution obtained in theorem 2.12 on the subspace of Dirac spinors $\mathbb{S}_A$.

Proof. : Let q be a point in $\mathcal{J}^+(p_0)$. Using proposition 2.3, proposition 2.7 and proposition 2.10, we have:

$$\begin{aligned} u(q) = &\not{D}^q\left(\int_{\sigma(q)} \left(\hat{\nabla}^p\Gamma_0 \cdot u, \tilde{U}_p\right)_p \frac{1}{4r_0 r}\mu_{\sigma(q)}\right) + \int_{\sigma(q)} \hat{\nabla}^q r_0 \cdot \left(\nabla^p\Gamma_0 \cdot u, \tilde{V}_p(q)\right)\frac{\mu_{\sigma(q)}(p)}{2r} \\ &+ \int_{\mathcal{D}(q)} (\hat{\nabla}^p\Gamma_0 \cdot u, \not{D}^q \tilde{V}_q)\mu_{\Gamma_0}(p). \end{aligned}$$

The bracket in the first integral can be calculated as follows:

$$\begin{aligned} \left(\tilde{U}_p, \frac{\hat{\nabla}^p\Gamma_0 \cdot u}{2rr_0}\right)_p &= k_p(q)\left(\tau_p(q), \frac{1}{2r}n \cdot u\right)_p \\ &= i\sqrt{2}k_p(q)\left(\tau_p(q), \frac{1}{2r}\left(-l^{AA'}\phi_A + l_{AA'}\psi^{A'}\right)\right)_p \\ &= i\sqrt{2}k_p(q)\left(\tau_p(q), \frac{1}{2r}\left(-o^A\bar{o}^{A'}\phi_A + o_A\bar{o}_{A'}\psi^{A'}\right)\right)_p \\ &= i\sqrt{2}k_p(q)\left(\tau_p(q), \frac{1}{2r}\left(-\phi_0\bar{o}^{A'} + \psi^{1'}o_A\right)\right)_p \\ &= i\sqrt{2}k_p(q)\frac{1}{2r}\left(-\phi_0(p)\bar{o}^{A'}(q) + \psi^{1'}(p)o_A(q)\right) \end{aligned}$$

We then use proposition 2.9 to calculate the first integral:

$$\begin{aligned}
&\not{D}^q\left(\int_{\sigma(q)}\left(\tilde{U}_p,\hat{\nabla}^p\Gamma_0\cdot u\right)_p\frac{1}{4r_0r}\mu_{\sigma(q)}\right)\\
&=i\sqrt{2}\not{D}^q\left(\int_{\sigma(q)}k_p(q)\left(-\phi_0(p)\overline{o}^{A'}(q)+\psi^{1'}(p)o_A(q)\right)\frac{1}{2r}\mu_{\sigma(q)}\right)\\
&=i\sqrt{2}\int_{\sigma(q)}\not{D}^q k_p(q)\left(\frac{-\phi_0(p)\overline{o}^{A'}(q)}{2r}+\frac{\psi^{1'}(p)o_A(q)}{2r}\right)\\
&+\hat{\nabla}^q r_0\cdot\nabla_l^p k_p(q)\left(\frac{-\phi_0(p)\overline{o}^{A'}(q)}{2r}+\frac{\psi^{1'}(p)o_A(q)}{2r}\right)\\
&-\frac{1}{r}\rho\hat{\nabla}^q r_0\cdot\left(-\phi_0(p)\overline{o}^{A'}(q)+\psi^{1'}(p)o_A(q)\right)\mu_{\sigma(q)}(p).
\end{aligned}$$

In order to simplify the calculation, the previous formula is projected on $\mathbb{S}_A$. The singular part on this subspace is written, after expansion:

$$\begin{aligned}
A &= i\sqrt{2}\int_{\sigma(q)}i\sqrt{2}\nabla^q_{AA'}\left(k_p(q)\left(\frac{-\phi_0(p)\overline{o}^{A'}(q)}{2r}\right)\right)\\
&\quad +i\sqrt{2}\nabla^q_{AA'}r_0\nabla^p_l\left(k_p(q)\left(\frac{-\phi_0(p)\overline{o}^{A'}(q)}{2r}\right)\right)\\
&\quad -(i\sqrt{2})k_p(q)\rho\nabla^q_{AA'}r_0\left(\frac{-\phi_0(p)\overline{o}^{A'}(q)}{r}\right)\mu_{\sigma(q)}(p).\\
&= -\int_{\sigma(q)}\nabla^q_{AA'}\left(k_p(q)\left(\frac{\phi_0(p)\overline{o}^{A'}(q)}{r}\right)\right)+\nabla^q_{AA'}r_0\nabla^p_l\left(k_p(q)\left(\frac{\phi_0(p)\overline{o}^{A'}(q)}{r}\right)\right)\\
&\quad -2k_p(q)\rho\nabla^q_{AA'}r_0\left(\frac{-\phi_0(p)\overline{o}^{A'}(q)}{r}\right)\mu_{\sigma(q)}(p).\\
&= -\int_{\sigma(q)}\phi_0(p)\nabla^q_{AA'}\left(k_p(q)\left(\frac{\overline{o}^{A'}(q)}{r}\right)\right)+\nabla^q_{AA'}r_0\nabla^p_l\left(k_p(q)\left(\frac{\phi_0(p)\overline{o}^{A'}(q)}{r}\right)\right)\\
&\quad -2k_p(q)\rho\nabla^q_{AA'}r_0\left(\frac{-\phi_0(p)\overline{o}^{A'}(q)}{r}\right)\mu_{\sigma(q)}(p).
\end{aligned}$$

Expanding all the products:

$$\begin{aligned}
A &= -\int_{\sigma(q)} \left(\phi_0(p)\left(\overline{o}^{A'}(q)\nabla^q_{AA'}\left(\frac{k_p(q)}{r}\right)+\left(\frac{k_p(q)}{r}\right)\nabla^q_{AA'}\overline{o}^{A'}(q)\right)\right. \\
&\quad+\left(\left(\frac{k_p(q)}{r}\right)\nabla^p_l\phi_0+\nabla^p_l\left(\frac{k_p(q)}{r}\right)\phi_0\right)\left(\nabla^q_{AA'}r_0\right)\overline{o}^{A'}(q) \\
&\quad\left.-2\left(\frac{k_q(p)}{r}\right)\rho(\nabla^q_{AA'}r_0)\phi_0(p)\overline{o}^{A'}(q)\right)\mu_{\sigma(q)}(p) \\
&= -\int_{\sigma(q)}\left(\frac{k_p(q)}{r}\right)\left(\nabla^p_l\phi_0(p)-2\rho\phi_0(p)\right)\left(\nabla^q_{AA'}r_0\right)\overline{o}^{A'}(q)\mu_{\sigma(q)}(p) \\
&\quad-\int_{\sigma(q)}\phi_0(p)\nabla_{AA'}\left(\left(\frac{k_p(q)}{r}\right)\overline{o}^{A'}(q)\right)\mu_{\sigma(q)}(p).
\end{aligned}$$

Since the quantities which appear in the integral are the restriction of u and its tangential derivative along the null cone $\mathcal{C}^+(p_0)$, ϕ can be replaced in the integral by the data of the Goursat problem ζ_0:

$$\begin{aligned}
&\int_{\sigma(q)}\left(\frac{k_p(q)}{r}\right)\left(\nabla^p_l\zeta_0(p)-2\rho\zeta_0(p)\right)\left(\nabla^q_{AA'}r_0\right)\overline{o}^{A'}(q)\mu_{\sigma(q)}(p) \\
&\qquad+\int_{\sigma(q)}\zeta_0(p)\nabla_{AA'}\left(\left(\frac{k_p(q)}{r}\right)\overline{o}^{A'}(q)\right)\mu_{\sigma(q)}(p).
\end{aligned}$$

We obtain the complete formula for Dirac spinors by adding the corresponding quantity on $\mathbb{S}^{A'}$, meaning:

$$\begin{aligned}
&\int_{\sigma(q)}\left(\frac{k_p(q)}{r}\right)\left(\nabla^p_l\xi^{1'}0(p)-2\rho\xi^{1'}(p)\right)\left(\nabla^{AA'}_q r_0\right)o_A(q)\mu_{\sigma(q)}(p) \\
&\qquad+\int_{\sigma(q)}\phi_0(p)\nabla^{AA'}\left(\left(\frac{k_p(q)}{r}\right)o_A(q)\right)\mu_{\sigma(q)}(p).
\end{aligned}$$

Noticing that the calculation has been done for $(\tilde{U}_p(q),\hat{\nabla}\Gamma_0\cdot u)$ in order to use the definition of $\tau(p,q)$, we obtain the complete formula using the antisymmetry of the symplectic product.♦

It is now possible to obtain the formula established by Penrose in [78] in the Minkowski case:

Theorem 2.14 (Penrose). *Let u be a solution of* (2.11).
Then, for all q in $\mathcal{J}^+(p_0)$, u can be written:

$$\begin{aligned}
u(q) &= \int_{\sigma(q)}\frac{1}{2\pi r}\left(\nabla^p_l\xi^{1'}0(p)-2\rho\xi^{1'}(p)\right)\left(\nabla^{AA'}_q r_0\right)o_A(q)\mu_{\sigma(q)}(p) \\
&\quad+\int_{\sigma(q)}\frac{1}{2\pi r}\left(\nabla^p_l\zeta_0(p)-2\rho\zeta_0(p)\right)\left(\nabla^q_{AA'}r_0\right)\overline{o}^{A'}(q)\mu_{\sigma(q)}(p)
\end{aligned}$$

Remark 2.15. *First of all, the meaning in the context of a flat space of the choice of the basis constructed in the previous section should be made precise:*

- *the spinor o^A is chosen to be constant on the null generators of the cone; the affine parameter r_0 is measured with respect to the vector $l^a = o^A\overline{o}^{A'}$;*

- *a direction on the cone* $\mathcal{C}^+(p_0)$ *being given together with a point* q *in* $\mathcal{J}^+(p_0)$*, let* p *be the intersection of* $\mathcal{C}^-(q)$ *with this direction on the null cone from* p_0*; the spinor* ι^A *is chosen so that* $n^a = \iota^A\bar{\iota}^{A'}$ *is colinear to the vector* $\vec{pq}$ *and satisfies:* $o_A\iota^A = 1$*; the affine parameter* r *is measured with respect to the vector* $n^a = \iota^A\bar{\iota}^{A'}$*;*

- *the basis is completed by the two vectors* $m^a = o^A\bar{\iota}^{A'}$ *and* $\overline{m}^a = \iota^A\bar{o}^{A'}$*.*

This construction is the "flat" version of the one made using parallel transport.

Proof. : As done in [78], it is sufficient to remark, for a direction ω on the cone $\mathcal{C}^+(p_0)$:

$$q = p_0 + r_0 l^a(\omega) + r n^a(q,\omega),$$

which implies:

$$\nabla^q r = l^a, \nabla^q r_0 = n^a$$

and $k_p(q) = \frac{1}{2\pi}$.♦

Remark 2.16. *It is interesting to note that the term that carries the curvature information in the singular part is:*

$$\nabla^q \left(\left(\frac{k_p(q)}{r} \right) o^A \right). \tag{2.13}$$

It is somehow difficult to give a precise geometric interpretation to equation (2.13)*. Nevertheless, clues can be found in theorem* 4.2.2 *in [39] that states that* $(k/r)^2$ *measures the growth rate of the measure* $\mu_{\sigma(q)}$*.*

3 Generalization to higher spin

In this section, we obtain an integral formula for solutions of the Goursat problem for the Dirac equation with arbitrary spin. The derivation of the formula is based on the representation formula for the Weyl equation which can be extracted from theorem 2.12.

Let us consider the characteristic Cauchy problem for spin $\frac{n}{2} = s \geq 1$ (n being the number of indices of a spinor):

$$\left\{ \begin{array}{lcl} \nabla^{AA'} u_{AB\ldots F} & = & 0 \text{ on } \mathcal{J}^+(p_0) \\ u_{00\ldots 0} & = & \theta_{00\ldots 0} \text{ on } \mathcal{C}^+(p_0) \end{array} \right. , \tag{3.1}$$

where $u_{AB\ldots F}$ satisfies the symmetry conditions:

$$u_{AB\ldots F} = u_{(AB\ldots F)}.$$

First of all, it must be noted that, on an arbitrary curved space, the problem (3.1) cannot be set if a consistency condition on the conformal curvature is not satisfied ([8], [37] and [70] for the Rarita-Schwinger case for a treatment of the Cauchy problem). It is known that for the Dirac massless equation for low spin ($n \leq 1$, i.e. scalar wave, Dirac-Weyl and Maxwell equations) this condition is always satisfied. For higher spin, it is satisfied whenever the space-time is conformally flat. Nonetheless, it is expected that the method could be adapted to the Rarita-Schwinger case which requires the space-time to be Ricci flat.

3.1 Generalization of Dirac equation to higher spin.

The construction that was made before for Dirac spinors is adapted here to spinors of higher valence so that the symmetry conditions of the Clifford multiplication and Dirac operator still hold.

Let us consider $\mathbb{E}$ the fibre bundle defined by:

$$\mathbb{E} = \mathbb{S}_{AB\dots F} \oplus \mathbb{S}^{A'}{}_{G\dots I}.$$

This fibre bundle is equipped with the symplectic product obtained from ε:

$$\varepsilon^{A\overline{A}}\varepsilon^{B\overline{B}}\dots\varepsilon^{F\overline{F}} \oplus \varepsilon_{A'\overline{A}'}\varepsilon^{G\overline{G}}\dots\varepsilon^{I\overline{I}}$$

and a Clifford multiplication by vectors: if $u = \phi_{AB\dots F} + \psi^{A'}{}_{G\dots I}$ belongs to $\mathbb{E}$, we define $e_{\mathbf{a}} \cdot u$, where $(e_{\mathbf{a}})_{\mathbf{a}=0,\dots,3}$ is the basis constructed in subsection 1.1:

$$e_{\mathbf{a}} \cdot u = -i\sqrt{2}g^{\mathbf{a}AA'}\phi_{AB\dots F} + i\sqrt{2}g^{\mathbf{a}}{}_{AA'}\psi^{A'}{}_{G\dots I}.$$

We finally define on smooth sections $u = \phi_{AB\dots F} + \psi^{A'}{}_{B\dots F}$ of $\mathbb{E}$ the following operator (that will be denoted by $\not{D}$ as the Dirac operator for Dirac spinors):

$$\not{D}u = i\sqrt{2}\big(- \nabla^{AA'}\phi_{AB\dots F} + \nabla_{AA'}\psi^{A'}{}_{G\dots I}\big). \tag{3.2}$$

The distributions on smooth sections of $\mathbb{E}$ are defined using the (non degenerate) symplectic product ε in the same way as in section 1.2. The duality bracket will still be denoted by $(,)_{\mathcal{D}'(\mathbb{E}),\mathcal{D}(\mathbb{E}))}$. Let $u = \phi_{AB\dots F} + \psi^{A'}{}_{G\dots I}$ and $v = \xi_{AB\dots F} + \zeta^{A'}{}_{G\dots I}$ be two smooth sections of $\mathbb{E}$. We have:

$$\begin{aligned}(v,u)_{\mathcal{D}'(\mathbb{E}),\mathcal{D}(\mathbb{E}))} &= \int_\Omega \Big(\varepsilon^{A\overline{A}}\varepsilon^{B\overline{B}}\dots\varepsilon^{F\overline{F}}\xi_{AB\dots F}\phi_{\overline{AB\dots F}} + \varepsilon_{A'\overline{A}'}\varepsilon^{G\overline{G}}\dots\varepsilon^{I\overline{I}}\zeta^{A'}{}_{G\dots I}\psi^{\overline{A}'}{}_{\overline{G\dots I}}\Big)\,\mu \\ &= \int_\Omega \xi_{A\dots F}\phi^{A\dots F} + \zeta^{A'}{}_{G\dots I}\psi_{A'}{}^{G\dots I}\mu\end{aligned}$$

We finally extend the Schrödinger-Lichnerowicz formula to arbitrary spin:

Proposition 3.1 (Schrödinger-Lichnerowicz formula for arbitrary spin)**.** *Let be $\psi_{F\dots I}$ a smooth section of $\mathbb{S}_{F\dots I}$ (n indices). Then the following formula holds:*

$$\begin{aligned}\nabla_{BA'}\nabla^{FA'}\psi_{F\dots I} &= \frac{1}{2}\nabla_{HH'}\nabla^{HH'}\psi_{BG\dots I} \\ &- X_B{}^F{}_F{}^D\psi_{DG\dots I} - X_B{}^F{}_G{}^D\psi_{FD\dots I} - \dots - X_B{}^F{}_I{}^D\psi_{FG\dots D}\end{aligned}$$

where X_{ABCD} is the curvature spinor:

$$X_{ABCD} = \frac{1}{4}R_{AX'B}{}^{X'}{}_{CY'D}{}^{Y'},$$

$R = R_{abcd}$ being the Riemann curvature tensor of Ω.

Remark 3.2. *It must be noted that the potential of the operator $\not{D}^2$, though linear, is no longer scalar and not even symmetric.*

Proof. : the proof is almost the same as the proof of proposition 1.37:

$$\begin{aligned}
\nabla_{BA'}\nabla^{FA'}\psi_{F\ldots I} &= \varepsilon^{FC}\nabla_{BA'}\nabla_C^{A'}\psi_{F\ldots I}\\
&= \varepsilon^{FC}\left(\nabla_{[B|A'}\nabla_{|C]}^{A'}\psi_{F\ldots I}+\nabla_{(B|A'}\nabla_{|C)}^{A'}\psi_{F\ldots I}\right)\\
&= \frac{1}{2}\varepsilon^{FC}\nabla_{HH'}\nabla^{HH'}\varepsilon_{BC}\psi_{F\ldots I}+\varepsilon^{FC}\nabla_{[B|A'}\nabla_{|C]}^{A'}\psi_{F\ldots I}\\
&= \frac{1}{2}\nabla_{HH'}\nabla^{HH'}\psi_{BG\ldots I}+\varepsilon^{FC}\nabla_{[B|A'}\nabla_{|C]}^{A'}\psi_{F\ldots I}
\end{aligned}$$

The spinor $\psi_{F\ldots I}$ is then split as the sum of tensor products of spinors of valence $\frac{1}{2}$, and, then as explained in [79] (vol. 1 p. 245, together with formula (4.9.4), (4.9.5) and (4.9.8)), using the fact:

$$\nabla_{[B|A'}\nabla_{|C]}^{A'}u_D = -X_{BC}{}^{E}{}_{D}u_E$$

for any smooth section of $\mathbb{S}_D$ (formula (4.9.8) in [79]), we obtain:

$$\nabla_{[B|A'}\nabla_{|C]}^{A'}\psi_{F\ldots I} = -X_{BCF}{}^{D}\psi_{DG\ldots I}-X_{BCG}{}^{D}\psi_{FD\ldots I}-X_{BCI}{}^{D}\psi_{FG\ldots D}$$

and finally:

$$\begin{aligned}
\nabla_{BA'}\nabla^{FA'}\psi_{F\ldots I} &= -\frac{1}{2}\nabla_{HH'}\nabla^{HH'}\psi_{F\ldots I}\\
&- \varepsilon^{FC}\left(X_{BCF}{}^{D}\psi_{DG\ldots I}-X_{BCG}{}^{D}\psi_{FD\ldots I}-X_{BCI}{}^{D}\psi_{FG\ldots D}\right)\\
&= -\frac{1}{2}\nabla_{HH'}\nabla^{HH'}\psi_{F\ldots I}\\
&- X_B{}^F{}_F{}^D\psi_{DG\ldots I}-X_B{}^F{}_G{}^D\psi_{FD\ldots I}-\cdots-X_B{}^F{}_I{}^D\psi_{FG\ldots D}\blacklozenge
\end{aligned}$$

As an obvious consequence of the definitions chosen for the Clifford multiplication and the Dirac operator on $\mathbb{E}$, the following proposition holds:

Proposition 3.3. *The Dirac operator $\not{D}$ on $\mathbb{E}$ and the Clifford multiplication by a vector field v on Ω are respectively symmetric and skew symmetric with respect to the duality bracket $(,)_{\mathcal{D}'(\mathbb{E}),\mathcal{D}(\mathbb{E})}$ that is to say, for any ϕ and ψ smooth sections of $\mathbb{E}$ with compact support :*

$$(\phi,\not{D}\psi)_{\mathcal{D}'(\mathbb{E}),\mathcal{D}(\mathbb{E}))} = (\not{D}\phi,\psi)_{\mathcal{D}'(\mathbb{E}),\mathcal{D}(\mathbb{E}))} \text{ and } (\phi,v\cdot\psi)_{\mathcal{D}'(\mathbb{E}),\mathcal{D}(\mathbb{E}))} = -(v\cdot\phi,\psi)_{\mathcal{D}'(\mathbb{E}),\mathcal{D}(\mathbb{E}))}.$$

All the methods that were developed for Dirac spinors can be used here, provided that we assume that we are working with $\mathbb{E}$-valued distributions. The structure of the fundamental solutions for the wave equation are the same:

$$\tilde{G}_q^{\pm}(p) = \kappa_q^{\pm}(p)\tau_p(q)\delta(\Gamma_q(p))+V_q(p)H_q^{\pm}(p).$$

where $\tilde{G}^{\pm}$ is a bidistribution in $\mathcal{E}(\mathbb{E})\boxtimes\mathcal{D}'(\mathbb{E})$ which satisfies the wave equation:

$$(\not{D}^p)^2G_q^{\pm}(p) = \overline{\delta}_p(q),$$

$\overline{\delta}_p(q)$ being the Dirac mass in p. The application $\tau_p(q)$ satisfies the equation:

$$(\tau_p(p),\phi) = \phi \text{ and } \nabla_i^q\Gamma_p(q)\nabla^i\tau_p(q) = 0. \tag{3.3}$$

Remark 3.4. *The functions τ and V are more complex to write and we do not even try to do so, since the properties given by the equations (3.3) are sufficient to conclude.*

A direct consequence of the previous remark is the following proposition:

Proposition 3.5. *The Dirac operator $\not{D}$ acting on sections of the fibre bundle $\mathbb{E}$ admits two fundamental solutions $G^{\pm}_p(q) = \not{D}^q \tilde{G}^{\pm}_p(q)$, in $\mathcal{D}'(\mathbb{E}) \boxtimes \mathcal{D}'(\mathbb{E})$, with, respectively, support in $\mathcal{C}^{\pm}(p)$, for any given p, which satisfy, in the sense of distributions:*

$$\not{D}^q G^{\pm}_p(q) = \overline{\delta}_p(q).$$

Finally, we present the compacted spin coefficient formalism introduced by Penrose and Rindler in [79]. Let o^A, ι^A be a given normalized spinor basis and consider the rescaling, for λ in $\mathbb{C}$:

$$o^A \longmapsto \lambda o^A, \iota^A \longmapsto \frac{\iota^A}{\lambda}. \tag{3.4}$$

Definition 3.6. *A spinor ϕ is said to be of weight (p, q) if and only if, under the transformation (3.4), ϕ is rescaled as:*

$$\phi \longmapsto \lambda^p \overline{\lambda}^q \phi$$

The integer $\frac{1}{2}(p-q)$ is the spin-weight of ϕ and $\frac{1}{2}(p+q)$ is its boost-weight.

We consider the Newman-Penrose tetrad $(l, n, m, \overline{m})$ associated with o^A, ι^A. We define the differential operators with regard to these weights: let ϕ be a (p, q) spinor. We define:

$$\begin{aligned} \text{þ}\phi &= \nabla_l \phi - p\epsilon\phi - q\bar{\epsilon}\phi \\ \eth'\phi &= \nabla_{\overline{m}}\phi - p\alpha\phi + q\overline{\alpha}\phi \end{aligned}$$

where $\epsilon = \iota^A \nabla_l o_A$ and $\alpha = \iota^A \nabla_{\overline{m}} o_A$.

Though the formalism of the Newman-Penrose tetrad will still be used, the usual notations o^A, ι^A for the basis spin basis are replaced by $\varepsilon_0^A, \varepsilon_1^A$. All the calculations will be performed using these notations. We must recall what is the link between these two notations: the spinor basis (o^A, ι^A) is rewritten $(\varepsilon_0^A, \varepsilon_1^A)$, so that the dual basis is $(\varepsilon_A^0, \varepsilon_A^1)$ with $\varepsilon_A^0 = -\iota_A$ et $\varepsilon_A^1 = o^A$. In this formalism, the spinors $\varepsilon_A^{\mathbf{I}}$ satisfy:

$$\varepsilon_A^{\mathbf{J}} \varepsilon_{\mathbf{I}}^A = \delta_{\mathbf{I}}^{\mathbf{J}}.$$

Let now consider the field equation for spin $\frac{n}{2}$:

$$\nabla^{AA'} \phi_{AB\ldots F} = 0$$

for a symmetric field $\phi_{AB\ldots F} = \phi_{(AB\ldots F)}$ with n indices; for j in $\{0, 1, \ldots, n\}$, we define:

$$\begin{aligned} \phi_j &= \underbrace{\varepsilon_0^A \ldots \varepsilon_0^C}_{\text{n-j times}} \underbrace{\varepsilon_1^D \ldots \varepsilon_1^F}_{\text{j times}} \phi_{AB\ldots F} \\ &= \underbrace{o^A \ldots o^C}_{\text{n-j times}} \underbrace{\iota^D \ldots \iota^F}_{\text{j times}} \phi_{AB\ldots F} \end{aligned}$$

which are the only relevant components to calculate the field $\phi_{AB\ldots F}$ wich can be written, because of its symmetry:

$$\begin{aligned} \phi_{A\ldots F} &= \sum_{j=0}^{n} \binom{n}{j} \phi_j \underbrace{\varepsilon^0_{(A} \ldots \varepsilon^0_C}_{\text{n-j times}} \underbrace{\varepsilon^1_D \ldots \varepsilon^1_{F)}}_{\text{j times}} \\ &= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \phi_j \underbrace{\iota_{(A} \ldots \iota_C}_{\text{n-j times}} \underbrace{o_D \ldots o_{F)}}_{\text{j times}} \end{aligned}$$

The quantity ϕ_j is a $(n-2r,0)$ scalar field. It is known to satisfy the following lemma (see [79], 4.12.42):

Lemma 3.7. *Let j be an integer in $\{2,\ldots,n-1\}$.*
Then ϕ_{j+1}, ϕ_j, ϕ_{j-1} and ϕ_{j-2} satisfy the following relation:

$$\mathfrak{p}\phi_j - \eth'\phi_{j-1} = (j-1)\sigma'\phi_{j-2} - j\tau'\phi_{j-1} + (n-j-1)\rho\phi_j - (n-j)\kappa\phi_{j+1}.$$

Remark 3.8. *This is the more accurate way to write down the constraints equations on the cone, since the restriction to the tangential derivatives is obvious.*

We conclude this section by giving the following relation between weighted scalars and differential forms (see [79], 4.14.70):

Proposition 3.9. *Let Σ be a two dimensional spacelike closed surface with volume form μ_Σ and α a $(1,-1)$ weighted spinor.*
Then the integral of $\eth'\alpha$ over Σ vanishes:

$$\int_\Sigma \eth'\alpha\mu_\Sigma = 0$$

3.2 Integral formula for spin $\frac{n}{2}$

Let us consider the future characteristic Cauchy problem for the Dirac operator on $\mathbb{E}$:

$$\left\{\begin{array}{lcl} \not{D}u & = & 0 \text{ on } \mathcal{J}^+(p_0) \\ u & = & \theta \text{ on } \mathcal{C}^+(p_0) \end{array}\right. , \tag{3.5}$$

where θ is a smooth compactly supported function on the cone $\int\mathcal{C}^+(p_0)$. It must be noted that the problem (3.5), contrary to the problem stated in (3.1), does not contain symmetry assumption. This assumption will be made afterwards to obtain the integral formula for (3.1).

By doing the same calculation as for proposition (2.2), a direct consequence of proposition (3.5) is the following integral formula:

Proposition 3.10. *Let u be a solution of (3.5) in $\mathbb{E}$. Then u can be written:*

$$u(q) = \not{D}^q\left(\int_{\sigma(q)}\left(\hat{\nabla}^p\Gamma_0\cdot u,\tilde{U}_p\right)_p\frac{1}{4r_0r}\mu_{\sigma(q)}\right) + \int_{\sigma(q)}\hat{\nabla}^q r_0\cdot\left(\nabla^p\Gamma_0\cdot u,\tilde{V}_p(q)\right)2r\mu_{\sigma(q)}(p)$$
$$+\int_{\mathcal{D}(q)}(\hat{\nabla}^p\Gamma_0\cdot u,\not{D}^q\tilde{V}_q)\mu_{\Gamma_0}(p).$$

The formula must now be simplified using the previous methods and a decomposition of the spinor u on the same basis as in subsection 2.2.1: u can be written:

$$u = \phi_{\mathbf{A}\ldots\mathbf{F}}\varepsilon_A^{\mathbf{A}}\ldots\varepsilon_F^{\mathbf{F}} + \psi^{\mathbf{A}'}{}_{\mathbf{F}\ldots\mathbf{F}}\varepsilon_{\mathbf{A}'}^{A'}\varepsilon_B^{\mathbf{B}}\ldots\varepsilon_F^{\mathbf{F}}.$$

The solution of

$$\left\{\begin{array}{lcl} \nabla^{AA'}u_{AB\ldots F} & = & 0 \text{ on } \mathcal{J}^+(p_0) \\ u_{AB\ldots F} & = & \theta_{AB\ldots F} \text{ on } \mathcal{C}^+(p_0) \end{array}\right. , \tag{3.6}$$

obtained by projecting on $\mathbb{S}_{A\ldots F}$ the integral formula given in proposition 3.10:

Proposition 3.11. *Let $u_{A\ldots F}$ be a solution of:*

$$\begin{cases} \nabla^{AA'}u_{AB\ldots F} &= 0 \ on\ \mathcal{J}^+(p_0) \\ u_{AB\ldots F} &= \theta_{AB\ldots F}\ on\ \mathcal{C}^+(p_0) \end{cases},$$

Then, $u_{A\ldots F}$ can be written:

$$\begin{aligned} u_{A\ldots F} = &\int_{\sigma(q)} \left(\frac{k_q(p)}{r}\right)\left(\nabla_l^p\phi_{0\boldsymbol{B}\ldots\boldsymbol{F}}(p) - 2\rho\phi_{0\boldsymbol{B}\ldots\boldsymbol{F}}(p)\right)\left(\nabla_{AA'}^q r_0\right)\varepsilon_0^{A'}(q)\varepsilon_B^{\boldsymbol{B}}(q)\ldots\varepsilon_F^{\boldsymbol{F}}(q)\mu_{\sigma(q)} \\ &+\int_{\sigma(q)}\phi_{0\boldsymbol{B}\ldots\boldsymbol{F}}(p)\nabla_{AA'}\left(\left(\frac{k_p(q)}{r}\right)\varepsilon_0^{A'}(q)\varepsilon_B^{\boldsymbol{B}}(q)\ldots\varepsilon_F^{\boldsymbol{F}}(q)\right)\mu_{\sigma(q)} \\ &+\int_{\sigma(q)}\hat{\nabla}^q r_0\cdot\left(\nabla^p\Gamma_0\cdot u,\tilde{V}_p(q)\right)\frac{\mu_{\sigma(q)}(p)}{2r}+\int_{\mathcal{V}_p}(\hat{\nabla}^p\Gamma_0\cdot u, \not{D}^p\tilde{V}_q)\mu_{\Gamma_0}(p) \end{aligned}$$

Remark 3.12. *Since our interest is in the singular part of the integral representation of the solution, we do not give a more explicit expression of the smooth part of the integral formula.*

Proof. The first step is to calculate the contraction $\hat{\nabla}^p\Gamma_0\cdot u$:

$$\begin{aligned} \hat{\nabla}^p\Gamma_0\cdot u &= -2i\sqrt{2}r_0\varepsilon_0^A\bar{\varepsilon}_0^{A'}(\phi_{\mathbf{A}\ldots\mathbf{F}}\varepsilon_A^{\mathrm{a}}\ldots\varepsilon_F^{\mathbf{F}}) + 2i\sqrt{2}r_0\varepsilon_A^0\bar{\varepsilon}_{A'}^0(\psi^{\mathbf{A}'}{}_{\mathbf{B}\ldots\mathbf{F}}\varepsilon_{\mathbf{A}'}^{A'}\varepsilon_B^{\mathbf{B}}\ldots\varepsilon_F^{\mathbf{F}}) \\ &= 2i\sqrt{2}r_0(-\varepsilon_0^A\bar{\varepsilon}_0^{A'}\phi_{\mathbf{A}\ldots\mathbf{F}}\varepsilon_A^a + \varepsilon_A^1\bar{\varepsilon}_{A'}^1\psi^{\mathbf{A}'}{}_{\mathbf{B}\ldots\mathbf{F}}\varepsilon_{\mathbf{A}'}^{A'})\varepsilon_B^{\mathbf{B}}\ldots\varepsilon_F^{\mathbf{F}} \\ &= 2i\sqrt{2}r_0(\phi_{0\mathbf{B}\ldots\mathbf{F}}\varepsilon_0^{A'} - \psi^{1'}{}_{\mathbf{B}\ldots\mathbf{F}}\varepsilon_A^1)\varepsilon_B^{\mathbf{B}}\ldots\varepsilon_F^{\mathbf{F}}. \end{aligned}$$

Since $\tau_p(q)$ is obtained by doing a tensor product between an element of the spin basis at a point p with the spinor obtained by parallely transporting this spinor along the geodesic from p to q, which is an element of the spin basis at q, the symplectic product $(\tau_p(q), \hat{\nabla}^p\Gamma_0\cdot u)$ realizes a switch between the variables p and q:

$$(\tau_p(q), \hat{\nabla}^p\Gamma_0\cdot u) = i\sqrt{2}r_0\left(\phi_{0\mathbf{B}\ldots\mathbf{F}}(p)\varepsilon_0^{A'}(q) - \psi^{1'}{}_{\mathbf{B}\ldots\mathbf{F}}(p)\varepsilon_A^1(q)\right)\varepsilon_B^{\mathbf{B}}(q)\ldots\varepsilon_F^{\mathbf{F}}(q).$$

The interversion of the symbols $\int$ and $\not{D}$ gives (we only make the calculation on $\mathbb{S}_{A\ldots F}$):

$$i\sqrt{2}\nabla_{AA'}^q\left(\int_{\sigma(q)}2i\sqrt{2}r_0\phi_{0\mathbf{B}\ldots\mathbf{F}}(p)\varepsilon_0^{A'}(q)\varepsilon_B^{\mathbf{B}}(q)\ldots\varepsilon_F^{\mathbf{F}}(q)\mu_{\Gamma_0,\Gamma_q}\right) \tag{3.7}$$

$$= -\int_{\sigma(q)}\phi_{0\mathbf{B}\ldots\mathbf{F}}(p)\nabla_{AA'}^q\left(\frac{k_q(p)}{r}\varepsilon_0^{A'}(q)\varepsilon_B^{\mathbf{B}}(q)\ldots\varepsilon_F^{\mathbf{F}}(q)\right)\mu_{\sigma(q)} \tag{3.8}$$

$$-\int_{\sigma(q)}\frac{k_q(p)}{r}(\nabla_l^p\phi_{0\mathbf{B}\ldots\mathbf{F}}(p))\left(\nabla_{AA'}^q r_0\right)\varepsilon_0^{A'}(q)\varepsilon_B^{\mathbf{B}}(q)\ldots\varepsilon_F^{\mathbf{F}}(q)\mu_{\sigma(q)} \tag{3.9}$$

$$-\int_{\sigma(q)}\nabla_l^p\left(\frac{k_q(p)}{r}\right)(\phi_{0\mathbf{B}\ldots\mathbf{F}}(p))\left(\nabla_{AA'}^q r_0\right)\varepsilon_0^{A'}(q)\varepsilon_B^{\mathbf{B}}(q)\ldots\varepsilon_F^{\mathbf{F}}(q)\mu_{\sigma(q)} \tag{3.10}$$

$$+\int_{\sigma(q)}2\rho\frac{k_q(p)}{r}(\phi_{0\mathbf{B}\ldots\mathbf{F}}(p))\left(\nabla_{AA'}^q r_0\right)\varepsilon_0^{A'}(q)\varepsilon_B^{\mathbf{B}}(q)\ldots\varepsilon_F^{\mathbf{F}}(q)\mu_{\sigma(q)} \tag{3.11}$$

which can be simplified in:

$$i\sqrt{2}\nabla^q_{AA'}\left(\int_{\sigma(q)} 2i\sqrt{2}r_0\phi_{0\mathbf{B}\ldots\mathbf{F}}(p)\varepsilon_0^{A'}(q)\varepsilon_B^{\mathbf{B}}(q)\ldots\varepsilon_F^{\mathbf{F}}(q)\mu_{\Gamma_0,\Gamma_q}\right) \tag{3.12}$$

$$= -\int_{\sigma(q)}\left(\frac{k_q(p)}{r}\right)\left(\nabla_l^p\phi_{0\mathbf{B}\ldots\mathbf{F}}(p) - 2\rho\phi_{0\mathbf{B}\ldots\mathbf{F}}(p)\right)\left(\nabla^q_{AA'}r_0\right)\varepsilon_0^{A'}(q)\varepsilon_B^{\mathbf{B}}(q)\ldots\varepsilon_F^{\mathbf{F}}(q)\mu_{\sigma(q)} \tag{3.13}$$

$$-\int_{\sigma(q)}\phi_{0\mathbf{B}\ldots\mathbf{F}}(p)\nabla_{AA'}\left(\left(\frac{k_q(p)}{r}\right)\varepsilon_0^{A'}(q)\varepsilon_B^{\mathbf{B}}(q)\ldots\varepsilon_F^{\mathbf{F}}(q)\right)\mu_{\sigma(q)} \tag{3.14}$$

The next part of the integral formula is exactly the same as in the case of the Weyl-Dirac spinors and is obtained in a similar way.♦

Finally, to obtain a solution of the full problem with symmetry, it is sufficient to symmetrize the unprimed indices in the formula; we then give a representation, when the problem (3.1) makes sense (i.e with adequat restrictions on the curvature for spin greater than $\frac{3}{2}$):

Theorem 3.13. *Let $u_{A\ldots F}$ be a solution of the symmetrized characterictic Cauchy problem*

$$\left\{\begin{array}{lcl}\nabla^{AA'}u_{AB\ldots F} & = & 0 \text{ on } \mathcal{J}^+(p_0)\\ u_0 & = & \theta_{AB\ldots F} \text{ on } \mathcal{C}^+(p_0)\end{array}\right., \tag{3.15}$$

where $u_{AB\ldots F}$ satisfies the symmetry conditions: $u_{AB\ldots F} = u_{(AB\ldots F)}$.
Then the singular part of the integral representation of $u_{A\ldots F}$, that is to say the part supported on the intersection of the cone is given by the formula:

$$\int_{\sigma(q)}\left(\frac{k_q(p)}{r}\right)\left(\nabla_l^p\phi_{0\boldsymbol{B}\ldots\boldsymbol{F}}(p) - 2\rho\phi_{0\boldsymbol{B}\ldots\boldsymbol{F}}(p)\right)\left(\nabla^q_{AA'}r_0\right)\varepsilon_0^{A'}(q)\varepsilon_B^{\boldsymbol{B}}(q)\ldots\varepsilon_F^{\boldsymbol{F}}(q)\mu_{\sigma(q)}$$

$$\int_{\sigma(q)}\phi_{0\boldsymbol{B}\ldots\boldsymbol{F}}(p)\nabla_{AA'}\left(\left(\frac{k_p(q)}{r}\right)\varepsilon_0^{A'}(q)\varepsilon_B^{\boldsymbol{B}}(q)\ldots\varepsilon_F^{\boldsymbol{F}}(q)\right)\mu_{\sigma(q)}$$

3.3 Integral formula for spin $\frac{n}{2}$ in the flat case

This subsection is devoted to the recovery of the Penrose formula; with the same notations as before, the following proposition holds:

Proposition 3.14 (Integral formula for the flat case for spin $\frac{n}{2}$)**.** *Let $\phi_{A\ldots F}$ be a solution of* (3.15) *on the Minkowski space time.*
Then ϕ can be written:

$$\phi_{A\ldots F} = (-1)^n\int_{\sigma(q)}(\nabla_l\phi_0 - (n+1)\rho\phi_0)\iota_A\ldots\iota_F\frac{\mu_{\sigma(q)}}{2\pi r}$$

Remark 3.15. *The formula which is given here agrees with the one obtained by Penrose in [78] (formula 4.9). The $(-1)^n$ comes from the fact that Penrose chooses the convention:*

$$\iota^A \longmapsto -\iota^A$$

because of the different choice of normalization (formula (4.7), op. cit.*):*

$$\iota_A o^A = 1$$

whereas our convention is:

$$o_A\iota^A = 1.$$

Proof. : We summarize the geometric elements required to perform the calculation:

Remark 3.16. *We recall the main properties of the spinor basis which was constructed in section 2.2.1:*

1. *the spinors o^A and ι^A are constant along a generator of the cone $\mathcal{J}^+(p_0)$, so that the spin coefficients corresponding to the derivatives of o^A, ι^A along the vector $l^a = o^A\bar{o}^A$, $\kappa, \varepsilon, \tau'$ are zero;*
2. *furthermore, for q in $\mathcal{J}^+(p_0)$, the basis (o^A, ι^A) is parallely transported along the integral curves of $\iota^A\bar{\iota}^{A'}$ and so, in the flat case, is constant along the null generators of the cone $\mathcal{J}^-(q)$;*
3. *the derivatives along m of o^A and ι^A are calculated explicitly (see [79], 4.12.28):*

$$\eth' o^A = -\rho\iota^A \text{ and } \eth'\iota^A = -\sigma' o^A;$$

4. *the derivatives of ι^A and r can be explicitly calculated by differentiating the relation:*

$$\vec{p_0 p}^a = r_0 l^a + r\iota^A\bar{\iota}^{A'}$$

for any p in $\mathcal{J}^+(p_0)$. Their derivatives are:

$$\nabla_{BB'}\iota^A = -\frac{1}{r}\iota_B\bar{o}_{B'}o^A \text{ and } \nabla_{AA'}r = o_A\bar{o}_{A'} \tag{3.16}$$

and, consequently, the only non-vanishing derivative of ι^A is

$$\nabla_m\iota^A = \frac{1}{r}o^A$$

and the spin coefficients

$$\tau' = -\iota^A\nabla_l\iota_A, \sigma' = -\iota^A\nabla_{\overline{m}}\iota_A, \beta' = -\alpha = -\iota^A\nabla_{\overline{m}}\iota_A \text{ and } \beta = -\alpha' = -\iota^A\nabla_m\iota_A$$

vanish.

5. *Using equations (3.16) and since ι^A is a $(-1,0)$-spinor and r is a $(1,1)$ scalar, the following derivatives vanish:*

$$\eth'\iota^A = 0 \text{ and } \eth' r = 0.$$

For the sake of clarity, the calculation is first performed for the Maxwell equations and then for the arbitrary spin. The first step is to write the Maxwell equations

$$\nabla^{AA'}\phi_{AB} = 0$$

as

$$\begin{array}{rcl} \nabla_l\phi_1 - \nabla_{\overline{m}}\phi_0 &=& (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2 \\ \nabla_l\phi_2 - \nabla_{\overline{m}}\phi_1 &=& -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\varepsilon)\phi_2 \\ \nabla_m\phi_1 - \nabla_n\phi_0 &=& (\mu - 2\gamma)\phi_0 + 2\tau\phi_1 - \sigma\phi_2 \\ \nabla_m\phi_2 - \nabla_n\phi_1 &=& -\nu\phi_0 + 2\mu\phi_1 + (\tau - 2\beta)\phi_2 \end{array} \tag{3.17}$$

with the convention $\phi_{00} = \phi_0$, $\phi_{10} = \phi_1$ and $\phi_{11} = \phi_2$. We then consider the singular part:

$$
\begin{aligned}
&\int_{\sigma(q)} \left(\frac{k_q(p)}{r}\right) \left(\nabla_l^p \phi_{0\mathbf{b}}(p) - 2\rho\phi_{0\mathbf{b}}(p)\right)\left(\nabla_{AA'}^q r_0\right)\varepsilon_0^{A'}\varepsilon_B^{\mathbf{b}}\mu_{\sigma(q)} \\
&= \int_{\sigma(q)} \left(\frac{k_q(p)}{r}\right) \left(\nabla_l^p \phi_{0\mathbf{b}}(p) - 2\rho\phi_{0\mathbf{b}}(p)\right)\left(\iota_A\iota_{A'}\right)\varepsilon_0^{A'}\varepsilon_B^{\mathbf{b}}\mu_{\sigma(q)} \\
&= -\int_{\sigma(q)} \left(\frac{k_q(p)}{r}\right) \left(\nabla_l^p \phi_{0\mathbf{b}}(p) - 2\rho\phi_{0\mathbf{b}}(p)\right)\iota_A\varepsilon_B^{\mathbf{b}}\mu_{\sigma(q)} \\
&= -\int_{\sigma(q)} \left(\frac{k_q(p)}{r}\right) \left(\nabla_l^p \phi_{00}(p) - 2\rho\phi_{00}(p)\right)\iota_A\iota_B\mu_{\sigma(q)} \\
&\quad \underbrace{-\int_{\sigma(q)} \left(\frac{k_q(p)}{r}\right) \left(\nabla_l^p \phi_{01}(p) - 2\rho\phi_{01}(p)\right)\iota_A o_B\mu_{\sigma(q)}}_{=B}
\end{aligned}
$$

with $\overline{\kappa} = \frac{1}{2\pi r}$. Using the first Maxwell equation (3.17), and since, for the choice of basis which was previously done, the spin coefficients $\kappa = o^A\nabla_l o_A$ and $\pi = -\iota^A\nabla_l\iota_A$ vanish, we obtain:

$$
\begin{aligned}
B &= \int_{\sigma(q)} \left(\frac{k_q(p)}{r}\right) \left(\nabla_l^p \phi_{01}(p) - 2\rho\phi_{01}(p)\right)\iota_A o_B\mu_{\sigma(q)} = \int_{\sigma(q)} \frac{1}{2\pi r}\left(\nabla_{\overline{m}}^p\phi_{00} - 2\alpha\phi_{00}\right)\iota_A o_B\mu_{\sigma(q)} \\
&= \int_{\sigma(q)} \nabla_{\overline{m}}\left(\frac{\phi_{00}\iota_A o_B}{2\pi r}\right) - 2\alpha\frac{\phi_{00}\iota_A o_B}{2\pi r} + \nabla_{\overline{m}} r\frac{\phi_{00}\iota_A o_B}{2\pi r^2} - \frac{\phi_{00}\iota_A\nabla_{\overline{m}}(o_B)}{2\pi r} - \frac{\phi_{00}\nabla_{\overline{m}}(\iota_A)o_B}{2\pi r}\mu_{\sigma(q)}
\end{aligned}
$$

Since

$$\nabla_{\overline{m}}^p r = \iota^B\overline{o}^{B'},\ \nabla_{BB'}^p r = \iota^B\overline{o}^{B'}o_B\overline{o}_{B'} \text{ and } \nabla_{\overline{m}}^p o_B = -\rho\iota_B,$$

and

$$\int_{\sigma(q)} \nabla_{\overline{m}}^p\left(\frac{\phi_{00}\iota_A o_B}{2\pi r}\right) - 2\alpha\frac{\phi_{00}\iota_A o_B}{2\pi r}\mu_{\sigma(q)} = 0,$$

by Stoke's theorem (cf. (4.14.70) in [79]; it is possible to reinterpret this expression using the compacted spin coefficient formalism), we obtain:

$$B = \int_{\sigma(q)} \rho\frac{\phi_{00}}{2\pi r}\iota_A\iota_B\mu_{\sigma(q)}.$$

We finally have the expected integral formula for the Maxwell equation:

$$
\begin{aligned}
&\int_{\sigma(q)} \left(\frac{k_q(p)}{r}\right) \left(\nabla_l^p \phi_{0\mathbf{b}}(p) - 2\rho\phi_{0\mathbf{b}}(p)\right)\left(\nabla_{AA'}^q r_0\right)\varepsilon_0^{A'}(q)\varepsilon_B^{\mathbf{b}}(q)\mu_{\sigma(q)} \\
&\qquad = \int_{\sigma(q)} (\nabla_l\phi_{00} - 2\rho)\iota_A\iota_B\frac{\mu_{\sigma(q)}}{2\pi r} - B \\
&\qquad = \int_{\sigma(q)} (\nabla_l^p\phi_{00} - 3\rho)\iota_A\iota_B\frac{\mu_{\sigma(q)}}{2\pi r}.
\end{aligned}
$$

The first step of the general proof is to notice that, as in the Maxwell case, the only remaining term in the flat case is the equation (3.13) since the term (3.14) vanishes. So the simplification of the equation (3.13) can be done using the same methods.

A direct consequence of remark 3.16 is that the relation given in lemma 3.7 is considerably simplified:

$$\forall j \in \{1, n-1\}, \mathfrak{p}\phi_j - 2\rho\phi_j = \eth'\phi_{j-1} + (n-j-1)\rho\phi_j. \tag{3.18}$$

In the Minkowski case, the only non-vanishing term in the integral formula is the following:

$$\int_{\sigma(q)} \left(\frac{k_q(p)}{r}\right) \left(\nabla_l^p \phi_{0\mathbf{b}\ldots\mathbf{f}}(p) - 2\rho\phi_{0\mathbf{b}\ldots\mathbf{f}}(p)\right)\left(\nabla^q_{(AA'} r_0\right)\varepsilon_0^{A'}(q)\varepsilon_B^{\mathbf{b}}(q)\ldots\varepsilon_{F)}^{\mathbf{f}}(q)\mu_{\sigma(q)}$$

which can be simplified in a flat space as:

$$-\int_{\sigma(q)} \frac{1}{2\pi r}\left(\nabla_l^p \phi_{0\mathbf{B}\ldots\mathbf{F}}(p) - 2\rho\phi_{0\mathbf{F}\ldots\mathbf{F}}(p)\right)\iota_{(A}(q)\varepsilon_B^{\mathbf{B}}(q)\ldots\varepsilon_{F)}^{\mathbf{F}}(q)\mu_{\sigma(q)}$$

Consider the generic term in this sum: let j be an integer in $\{1, \ldots, n-1\}$:

$$\int_{\sigma(q)} \frac{1}{2\pi r}(\nabla_l\phi_j - 2\rho)\iota_{(A}(q)\varepsilon_B^{\mathbf{B}}(q)\ldots\varepsilon_{F)}^{\mathbf{F}}(q)\mu_{\sigma(q)}$$

Since ϕ_j is obtained by contracting j times $\phi_{A\ldots F}$ with ι^A and $n-j$ times with o^A, it means that there is exactly j times o_A and $n-j-1$ times $-\iota_A$ in the list $\varepsilon_B^{\mathbf{B}}(q)\ldots\varepsilon_F^{\mathbf{F}}(q)$; since the sum is symmetric, it can be written:

$$\int_{\sigma(q)} \frac{1}{2\pi r}(\nabla_l\phi_j - 2\rho\phi_j)\underbrace{\iota_{(A}(-\iota_B)\ldots(-\iota_C)}_{n-j \text{ terms}}\underbrace{o_D\ldots o_{F)}}_{j \text{ terms}}\mu_{\sigma(q)}.$$

Using equation (3.18), it becomes:

$$\int_{\sigma(q)} \frac{1}{2\pi r}(\nabla_l\phi_j - 2\rho\phi_j)\iota_{(A}\ldots\iota_C o_D\ldots o_{F)}\mu_{\sigma(q)}$$
$$= \int_{\sigma(q)} \frac{1}{2\pi r}(n-j-1)\rho\phi_j\iota_{(A}\ldots\iota_C o_D\ldots o_{F)}\mu_{\sigma(q)} + \int_{\sigma(q)} \frac{1}{2\pi r}\eth'(\phi_{r-1})\iota_A\ldots\iota_C o_D\ldots o_{F)}\mu_{\sigma(q)}.$$

Using remarks 3.16, the last integral is written as a difference:

$$\int_{\sigma(q)} \frac{1}{2\pi r}\eth'(\phi_{j-1})\iota_{(A}\ldots\iota_C o_D\ldots o_{F)}\mu_{\sigma(q)} = \int_{\sigma(q)} \eth'\left(\frac{\phi_{j-1}\iota_{(A}\ldots\iota_C o_D\ldots o_{F)}}{2\pi r}\right)\mu_{\sigma(q)} \tag{3.19}$$

$$+j\int_{\sigma(q)} \rho\phi_{j-1}\underbrace{\iota_{(A}\ldots\iota_D}_{n-j-1}\underbrace{o_E\ldots\iota_{F)}}_{j-1}\frac{\mu_{\sigma(q)}}{2\pi r}. \tag{3.20}$$

It has already been noted that:

1. r is $(1,1)$ scalar;
2. ϕ_{j-1} is a $(n-2j+2, 0)$ scalar;
3. $\iota_{(A}\ldots\iota_C o_D\ldots o_{F)}$ is a $(2j-n, 0)$ spinor.

As a consequence, the term integrated in the left-hand side of equation (3.19) and under the derivation $ð'$ is $(1,-1)$ spinor. In order to apply lemma 3.9, this spinor is contracted with n constant arbitrary spinors; this gives:

$$\int_{\sigma(q)} ð' \left(\frac{\phi_{j-1}}{2\pi r} \underbrace{\iota_{(A} \dots \iota_C}_{n-j \text{ terms}} \underbrace{o_D \dots o_{F)}}_{j \text{ terms}} \right) \mu_{\sigma(q)} = 0.$$

Finally, we obtain:

$$\int_{\sigma(q)} (\nabla_l \phi_j - 2\rho\phi_r) \underbrace{\iota_{(A} \dots \iota_C}_{n-j \text{ terms}} \underbrace{o_D \dots o_{F)}}_{j \text{ terms}} \frac{\mu_{\sigma(q)}}{2\pi r} = \tag{3.21}$$

$$(n-j-1) \int_{\sigma(q)} \rho\phi_j \underbrace{\iota_{(A} \dots \iota_C}_{n-j} \underbrace{o_D \dots o_{F)}}_{j} \frac{\mu_{\sigma(q)}}{2\pi r} + j \int_{\sigma(q)} \rho\phi_{j-1} \underbrace{\iota_{(A} \dots \iota_D}_{n-j+1} \underbrace{o_E \dots \iota_{F)}}_{j-1} \frac{\mu_{\sigma(q)}}{2\pi r} \tag{3.22}$$

Theses terms are added to obtain the complete expression of the integral formula:

$$\begin{aligned}
&\int_{\sigma(q)} (\nabla_l \phi_{0\mathbf{b}\dots\mathbf{f}} - 2\rho) \iota_{(A}(q) \varepsilon^{\mathbf{b}}_B(q) \dots \varepsilon^{\mathbf{f}}_{F)}(q) \frac{\mu_{\sigma(q)}}{2\pi r} \\
&= \sum_{j=0}^{n-1} (-1)^{n-j-1} \binom{n-1}{j} \int_{\sigma(q)} (\nabla_l \phi_j - 2\rho\phi_j) \underbrace{\iota_{(A} \dots \iota_C}_{n-j \text{ terms}} \underbrace{o_D \dots o_{F)}}_{j \text{ terms}} \frac{\mu_{\sigma(q)}}{2\pi r} \\
&= \int_{\sigma(q)} (\nabla_l \phi_0 - 2\rho\phi_0) \iota_A \dots \iota_F \frac{\mu_{\sigma(q)}}{2\pi r} + \\
&\sum_{j=1}^{n-1} (-1)^{n-j-1} (n-j-1) \binom{n-1}{j} \int_{\sigma(q)} \rho\phi_j \underbrace{\iota_{(A} \dots \iota_C}_{n-j} \underbrace{o_D \dots o_{F)}}_{j} \frac{\mu_{\sigma(q)}}{2\pi r} \\
&+ \sum_{j=1}^{n-1} (-1)^{n-j-1} j \binom{n-1}{j} \int_{\sigma(q)} \phi_{j-1} \underbrace{\iota_{(A} \dots \iota_D}_{n-j-1} \underbrace{o_E \dots \iota_{F)}}_{j-1} \frac{\mu_{\sigma(q)}}{2\pi r}.
\end{aligned}$$

The sum is split in two and reindexed:

$$\begin{aligned}
&\sum_{j=1}^{n-1} (-1)^{n-j-1} (n-j-1) \binom{n-1}{j} \int_{\sigma(q)} \rho\phi_j \underbrace{\iota_{(A} \dots \iota_C}_{n-j} \underbrace{o_D \dots o_{F)}}_{j} \frac{\mu_{\sigma(q)}}{2\pi r} \\
&+ \sum_{j=1}^{n-1} (-1)^{n-j-1} j \binom{n-1}{j} \int_{\sigma(q)} \phi_{j-1} \underbrace{\iota_{(A} \dots \iota_D}_{n-j-1} \underbrace{o_E \dots \iota_{F)}}_{j-1} \frac{\mu_{\sigma(q)}}{2\pi r} = \\
&\sum_{j=1}^{n-2} (-1)^{n-j-1} \underbrace{\left((n-j-1)\binom{n-1}{j} - (j+1)\binom{n-1}{j+1} \right)}_{=0} \int_{\sigma(q)} \rho\phi_j \underbrace{\iota_{(A} \dots \iota_C}_{n-j} \underbrace{o_D \dots o_{F)}}_{j} \frac{\mu_{\sigma(q)}}{2\pi r}
\end{aligned}$$

The remaining terms are then:

$$(-1)^{n-1} \int_{\sigma(q)} (\nabla_l \phi_0 - 2\rho\phi_0) \iota_A \dots \iota_F \frac{\mu_{\sigma(q)}}{2\pi r} - (-1)^{n-1} \binom{n-1}{1} \int_{\sigma(q)} \rho\phi_0 \iota_A \dots \iota_F \frac{\mu_{\sigma(q)}}{2\pi r}$$

and the integral formula is, because of the antisymmetry of the symplectic product:

$$\phi_{A\ldots F} = (-1)^n \int_{\sigma(q)} (\nabla_l \phi_0 - (n+1)\rho\phi_0)\iota_A \ldots \iota_F \frac{\mu_{\sigma(q)}}{2\pi r} \tag{3.23}$$

is proved.♦

Concluding remarks

1. Klainerman-Rodnianski state in [60] that a C^2 metric (or a square-integrable Riemann curvature) is sufficient to write the singular part of the Kirchoff-Sobolev parametrix for the Einstein equations. It is expected that such a regularity will not prevent the use of this method for the arbitrary spin Dirac equation.

2. The construction of the representation formula is flexible enough to be used with other fiber bundles. Provided that the correct geometric hypotheses are stated for the manifold, such a representation can thus be obtained for the Rarita-Schwinger (or gravitino) equations.

3. Chrusciel-Shatah obtained in [18] L^2-estimates for the Yang-Mills equations. We hope that such estimates can be obtained using the Friedlander construction of an integral formula. Nonetheless, it must be noted that they intensively used the gauge freedom which exists for the Yang-Mills equation: they used both the Cronström gauge (to obtain pointwise estimates) and the temporal gauge (to obtain estimates on spacelike slices). Similar estimates for the Dirac equations could help to explain, for instance, the loss of regularity observed in the characteristic Cauchy problem in [48] (section 6).

This work was partially supported by the ANR project JC0546063 "Equations hyperboliques dans les espaces-temps de la relativité générale : Diffusion et résonances."

Chapter 2

Characteristic Cauchy problem for the nonlinear wave equation on a curved background

Résumé: L'objectif est d'établir l'existence d'un opérateur de scattering conforme pour une équation des ondes non linéaire. S'inspirant des travaux de Mason-Nicolas ([62, 63]) sur l'équation des ondes linéaires, on adapte ici une méthode utilisée par Hörmander ([53]) pour obtenir des estimations d'énergie afin d'établir l'existence d'opérateurs de trace sur l'infini conforme associés aux solutions de l'équation non linéaire considérée.

Introduction

Scattering theory is widely used to understand and describe the asymptotic behavior of solutions of evolution equations. As a consequence, it has a great importance in relativity to understand the influence of the geometry on the propagation of waves. Scattering in relativity was developed by many authors: Dimock ([25]), Dimock and Kay ([28, 26, 27]) and more recently, Bachelot ([1, 4, 2, 3]), Bachelot and Bachelot-Motet ([5]), Häfner ([47, 48]), Häfner-Nicolas ([50]), Melnyk ([64]) and Daudé [22, 23].

The scattering method used by these authors relies on spectral theory: this requires the metric to be static (or that it exists a timelike Killing vector field) which cannot be achieved on a generic spacetime. It has consequently been necessary to develop a scattering theory which is not time dependent. As remarked by Friedlander in [40], it is possible to use a conformal rescaling to study an asymptotic behavior. This method was used for the first time by Baez-Segal-Zhou in [6] for the wave equation on the Minkowsky space-time. Their method consisted in embedding conformally the Minkowski space-time in a bigger compact manifold.

The conformal compactification of a space-time was first introduced by Penrose in the sixties in [77] to describe the asymptotic behavior of solutions of the Dirac equations. A boundary, which represents in some way the infinity for causal curves, is added to the manifold. This boundary is divided into two connex components $\mathscr{I}^+$ and $\mathscr{I}^-$. When the spacetime satisfies the Einstein equations (with no cosmological constant), $\mathscr{I}^+$ and $\mathscr{I}^-$ are light cones from two singularities i^+ and i^-. The asymptotic behavior can then be obtained by considering the traces on these hypersurfaces of a solution of the conformal wave equation (more precisely the scattering operator is obtained from the trace operators on $\mathscr{I}^+$ and $\mathscr{I}^-$). Asymptotically simple curved spacetimes, that is to say spacetimes admitting a conformal compactification, with specifiable regularity at i^+ and i^-, were constructed by Chrusciel-Delay ([16, 17]), Corvino ([19]) and Corvino-Schoen ([20]). Mason and Nicolas successfully adapted the method of Baez-Segal-Zhou in the linear case for the Dirac and Dirac-Maxwell equation by in [62] on this curved background. They also obtained a compete peeling result for the wave equation on a Schwarzschild background in [63].

This paper presents the construction of a conformal scattering operator for the conformally invariant defocusing cubic wave equation:

$$\nabla_a \nabla^a \Phi + b\Phi^3 = 0$$

on the asymptotically simple space-time obtained by Chrusciel-Delay and Corvino-Schoen. Our construction relies on vector fields methods which were previously used to obtain the well-posedness of the Cauchy problem equation (see the result of Cagnac–Choquet-Bruhat in [11]) to obtain energy estimates. The choice which is made here is for the vector field is the same as the one made in [63] and the techniques are essentially the same as in [62].

A specific method is used to handle the singularity in i^+: the characteristic hypersurface is described as the graph of a function. This method was introduced by Hörmander in [53] and generalized in [73, 72] by Nicolas to establish the well-posedness of the characteristic Cauchy problem.

The main obstacle and difference with the linear case are the necessity to obtain uniform estimates of the non linearity. This requires to obtain uniform Sobolev embeddings from H^1 into L^6. This is achieved by considering results concerning the constant associated with the embeddings given by Stein ([81] for extension theorem) and Hébey ([51] for Sobolev embeddings).

The paper is organized as follows:

- the first section introduces the geometrical and analytical background: the space-times obtained by Corvino-Schoen and Chrusciel-Delay are precisely defined and the function space on the characteristic hypersurface at infinity are given.
- The a priori estimates are derived in section 2: these estimates are established in three specific subsets of M: a neighborhood of i^0 where the estimates come from the asymptotic behavior of the chosen vector field (the Morawetz vector field), a neighborhood of i^+ where the estimates are established by following the method developed by Hörmander, and finally in a neighborhood of a Cauchy hypersurface. The techniques consist essentially in the use of Gronwall lemma and Stokes theorem.
- Section 3 is devoted to the well-posedness for small data of the Cauchy problem. The proof is made as follows: estimates on the propagator of the cubic wave equation are established from uniform Sobolev estimates. Using a contraction result, a local existence theorem for the characteristic problem is then obtained for small data: the solution is constructed up to a uniformly spacelike hypersurface close enough to the conformal infinity. Finally, a Cauchy problem from this hypersurface gives a global solution.
- Finally, we prove in section 4 the existence of a Lipschitz conformal scattering operator obtained from two trace operators.
- Section 5 introduces another approach for the a priori estimates based on a weakly spacelike foliation. This part of the work remains unachieved because it requires a control of the Killing form associated with the foliation. The author has not yet been able to obtain it.

Conventions and notation

Let (M, g) be a 4 dimensional manifold of Chrusciel-Delay/ Corvino-Schoen type. Its compactification is denoted by $(\hat{M}, \hat{g})$. The associated connections are denoted ∇ and $\hat{\nabla}$.

Let us consider on M the following nonlinear wave equation:

$$\nabla_a \nabla^a \Phi + b\Phi^3 = 0$$

We assume that:

1. b is positive;
2. b admits a continuous extension to $\hat{M}$ such that b vanishes at $\mathscr{I}$;
3. b satisfies: there exists a constant c such that, uniformly on $\hat{M}$:

$$\exists c, |\hat{T}^a \nabla_a b| \leq cb.$$

Remark 0.1. *1. The positivity of b corresponds to the defocusing case.*

2. The vanishing of b on $\mathscr{I}$ implies that the non linearity vanishes at infinity. This hypothesis is made so that we do not have to deal with Sobolev embeddings on $\mathscr{I}$.

3. *Since $\hat{M}$ is compact, the differential inequality satisfied by b does in fact not impose another specific asymptotic behaviour than the fact that $\hat{T}_a\nabla^a\phi$ decrease and vanishes at the same rate of b.*

The following notations will be used:

- we will note:

$$\phi \lesssim \psi$$

where ϕ and ψ are two functions over U, a subset of M, whenever there exists a constant C, depending only on the geometry, the vector $\hat{T}^a$, the Killing form $\hat{\nabla}^{(a}\hat{T}^{b)}$ and the function b, such as:

$$\psi \leq C\phi \text{ on } U.$$

If the ψ and ϕ both satisfy:

$$\phi \lesssim \psi \text{ and } \psi \lesssim \phi,$$

we say that ϕ and ψ are equivalent and note:

$$\phi \approx \psi.$$

- The geometric notations are the following:
  - The quantities with $\hat{}$ are geometric quantities related to the unphysical metric.
  - $\mu[\hat{g}]$ is the volume form associated with the metric $\hat{g}$.
  - If ν is a form over $\hat{M}$, then $\star\nu$ is its Hodge dual. If ν is a 1-form and V the vector field associated to ν via the metric $\hat{g}$, then:

$$\star\nu = V \lrcorner \mu[\hat{g}] \text{ or } \star V_a = V^a \lrcorner \mu[\hat{g}]$$

where $\lrcorner\mu[\hat{g}]$ is the contraction with the volume form $\mu[\hat{g}]$

  - i_Σ is the restriction to the submanifold Σ. The pull-forward of a form ν on $\hat{M}$ over the tangent space to Σ is denoted by $i^\star_\Sigma(\Sigma)$.

1 Functional and geometric preliminaries

We present in this section the geometric and analytic background to the present work. A specific care is brought to the structure at null infinity and the definition of function spaces on that structure.

1.1 Geometric framework

The geometric framework is based on the results of Corvino-Schoen ([20, 19]) and Chrusciel-Delay([16, 17]). They gave a construction of asymptotically simple spacetimes satisfying the Einstein equations with specifiable regularity at null and timelike infinities.

1.1.1 Asymptotic simplicity

The notion of asymptotically simple spacetimes was introduced by Penrose as a general model for asymptotically flat Einstein spacetimes and their conformal compactification (see [79] definition 9.6.11):

Definition 1.1. *A smooth Lorentzian manifold M satisfying the Einstein equations is said to be (C^k) asymptotically simple if there exists a smooth Lorentzian manifold $\hat{M}$ with boundary, a metric $\hat{g}$ and a conformal factor Ω such that:*

1. *M is the interior of $\hat{M}$;*
2. *$\hat{g} = \Omega^2 g$ in M;*
3. *$\hat{g}$ and Ω are C^k on $\hat{M}$;*
4. *Ω is positive in M; Ω vanishes at the boundary $\mathscr{I}$ of $\hat{M}$ and $d\Omega$ does not vanish at $\mathscr{I}$;*
5. *every null geodesic in M acquires a past and future end-point in $\mathscr{I}$*

We assume that the boundary $\mathscr{I}$ is C^2 (which is sufficient for this work). It is known that this boundary is a null hypersurface (that it is to say that the restriction of the metric to $\mathscr{I}$ is degenerate) provided that the cosmological constant is zero. Furthermore, $\mathscr{I}$ has two connected components $\mathscr{I}^+$ and $\mathscr{I}^-$ consisting of, respectively, the future and past endpoints of null geodesics. $\mathscr{I}^+$ and $\mathscr{I}^-$ are both diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$.

The manifold (M, g) is usually referred to as the physical space-time and its compactification is referred to as the unphysical space-time. In order to remain consistent with this notation all along this paper, the quantities associated with the unphysical metric are denoted with a "ˆ".

1.1.2 Global hyperbolicity

An important assumption in the context of the Cauchy problem for a wave equation is the possibility to write the equation as an evolution partial differential equation. This is usually achieved by requiring that the manifold M is globally hyperbolic:

Definition 1.2. *A Lorentzian manifold (M, g) is said to be globally hyperbolic if, and only if, there exists in M a global Cauchy hypersurface, i.e. a spacelike hypersurface such that any inextendible timelike curve intersects this surface exactly once.*

A useful consequence of this is the existence of a time function on M and the parallelizability of M, that is to say the existence of a global section of the principal bundle of orthonormal frames (see the work of Geroch in [44, 45] and Bernal-Sanchez in [7]).

In the case of an asymptotically simple manifold (M, g), this property extends of course to the manifold $(\hat{M}, \hat{g})$.

1.1.3 Corvino-Schoen/Chrusciel-Delay space-times

We can then introduce the spacetimes obtained by Corvino-Schoen and Chrusciel-Delay:

Definition 1.3. *A space-time M is of Chrusciel-Delay/Corvino-Schoen type if:*

1. *M is asymptotically simple;*

2. *M is globally hyperbolic; let Σ_0 be a spacelike Cauchy hypersurface; M is then diffeomorphic to $\mathbb{R} \times \Sigma_0$*

3. *$\hat{M}$ can be completed into a compact manifold by adding three points i^0, i^+ and i^- such that $\mathcal{I}^+$ and $\mathcal{I}^-$ are respectively the past null and future null cones from i^+ and i^- and i^0 is the conformal infinity of the spacelike hypersurface Σ_0 for the metric $\hat{g}$;*

4. *there exists a compact set K in $\Sigma 0$ such that $(\mathbb{R}\times\Sigma_0\backslash K, g)$ is isometric to $(\mathbb{R}\times]r_0, +\infty[\times\mathbb{S}^2, g_S)$, where g_S is the Schwarzschild metric with mass m and $r_0 > 2m$;*

5. *there exists a neighborhood of i^+ such that the metric $\hat{g}$ is obtained in this neighborhood as the restriction of a smooth (C^2) Lorentzian metric of an extension of $\hat{M}$ in the given neighborhood of i^+. The same property holds in i^-.*

Remark 1.4. 1. *The result of Corvino-Schoen/Chrusciel-Delay states that the metric is isometric to the Kerr metric outside a compact set; we restrict ourself to a Schwarzschild metric.*

2. *The extension of the manifold $\hat{M}$ in the neighborhood of i^+ was used by Mason-Nicolas (see [62, 63]). The point i^+ remains singular in $\hat{M}$ but it is nonetheless possible, because of the existence of this extension, to consider geometric data in i^+ (metric, exponential map, connection, curvature) as being the one obtained from the Lorentzian manifold extending $(\hat{M}, \hat{g})$ in the a neighborhood of this point.*

3. *The point i^0 is a "real" singularity of $\hat{M}$; nonetheless, the geometry is completely known in its neighborhood. This singularity is the main problem we have to deal with in this paper, being an obstacle to global estimates and Sobolev embeddings for instance.*

The last part of this section is devoted to the geometric description around the point i^0, that is to say in the Schwarzschild part of the manifold.

We consider here a neighborhood O in $\hat{M}$ of i^0 where the metric g is the Schwarzschild metric. This metric, in the Schwarzschild coordinates $(t, r, \omega_{\mathbb{S}^2})$, can be written:

$$g = F(r)\mathrm{d}t^2 - \frac{1}{F(r)}\mathrm{d}r^2 - r^2\mathrm{d}^2\omega_{\mathbb{S}^2}$$

where:

$$F(r) = 1 - \frac{2m}{r}$$

with m a positive constant. Introducing the new coordinates

$$r^* = r + 2m\log(r - 2m), u = t - r^* \text{ and } R = \frac{1}{r}.$$

The metric can then be expressed:

$$g = (1 - 2mR)\mathrm{d}^2u - \frac{2}{R^2}\mathrm{d}u\mathrm{d}R - \frac{1}{R^2}\mathrm{d}^2\omega_{\mathbb{S}^2}.$$

where $\mathrm{d}^2\omega_{\mathbb{S}^2}$ stands for the standard volume form on the 2-sphere, which can be written in polar coordinates (θ, ϕ):

$$\mathrm{d}^2\omega_{\mathbb{S}^2} = \sin\theta\mathrm{d}\theta \wedge \mathrm{d}\phi.$$

Its inverse is:

$$g^{-1} = \frac{-2}{R^2}\partial_u\partial_R - (1-2mR)\partial_R^2 - \frac{1}{R^2}\eth\eth'.$$

The part of the Cauchy hypersurface Σ_0 is then given by the equation $\{t=0\}$ in these coordinates.

In the neighborhood O, the conformal factor is chosen to be:

$$\Omega = R$$

and is extended smoothly in $M\backslash O$. The conformal metric is then:

$$\hat{g} = R^2(1-2mR)\mathrm{d}^2u - 2\mathrm{d}u\mathrm{d}R - \mathrm{d}^2\omega_{\mathbb{S}^2}.$$

and its inverse:

$$\hat{g}^{-1} = -2\partial_u\partial_R - R^2(1-2mR)\partial_R^2 - \eth\eth'.$$

The point i^0 is given in these coordinates by $u=-\infty$, $R=0$.

The description of the geometry around i^0 is completed by the following lemma (lemma A.1 in [63]):

Lemma 1.5. *Let $\epsilon>0$. There exists $u_0<0$, $|u_0|$ large enough, such that the following decay estimates in the coordinate (u,r,θ,ψ) hold:*

$$r<r^*<(1+\epsilon)r, 1<Rr^*<1+\epsilon, 0<R|u|<1+\epsilon,$$
$$1-\epsilon<1-2mR<1, 0<s=\frac{|u|}{r^*}<1$$

Proof. The proof of this lemma is straightforward: it only consists in writing simultaneously the asymptotic behavior or the continuity over $\hat{M}$ of each of the coordinates involved in the lemma. ⧫

Remark 1.6. *1. As mentioned in the introduction, we choose to work in the neighborhood of i^0 with the Morawetz vector $\hat{T}^a$ defined by:*

$$\hat{T}^a = u^2\partial_u - 2(1+uR)\partial_R.$$

The squared norm of this vector is:

$$\hat{g}_{ab}\hat{T}^a\hat{T}^b = u^2(4(1+uR)+u^2R^2(1-2mR)).$$

This polynomial in uR vanishes at:

$$2\frac{1\pm\sqrt{2mR}}{1-2mR} = \frac{-2}{1\mp\sqrt{2mR}},$$

so that, if R is chosen small enough, these roots are arbitrarily close to -2. Let ϵ be a positive number chosen such that the inequalities in lemma 1.5 hold for a given u_0. The larger zero of this polynomial satisfies:

$$\frac{-2}{1+\sqrt{2mR}} \leq \frac{-2}{1+\sqrt{\epsilon}}.$$

Choosing ϵ such that

$$\frac{-2}{1+\sqrt{\epsilon}} \leq -1-\epsilon,$$

the norm of $\hat{T}^a$ is then uniformly controlled by:

$$\hat{g}_{ab}\hat{T}^a\hat{T}^b = u^2(4(1+uR) + u^2R^2(1-2mR)) \geq 4u_0^2\epsilon$$

on the domain $\Omega^+_{u_0} = \{(u, R, \omega_{\mathbb{S}^2})|u < u_0\} \cap J^+(\Sigma_0)$, $J^+(\Sigma_0)$ being the future of Σ_0.

2. *Another criterion, given in proposition 2.5, will be required to define ϵ.*

1.2 Analytical requirements

We introduce in this section the technical and analytical tools which are required to the description of the solution for the wave equation.

1.2.1 Conformal wave equation

We recall here how the problem on the physical space time is transformed into a problem on the unphysical space-time. This is based on the classic transformation of the wave d'Alembertian operator:

Lemma 1.7. *Let M a Lorentzian manifold endowed with the conformal metrics g and $\hat{g} = \Omega^2 g$ where Ω is a conformal factor in $C^2(M)$. The connections associated with g and $\hat{g}$ are denoted by ∇ and $\hat{\nabla}$ respectively.*
Then, for any smooth function ϕ on M, the following equality holds:

$$\nabla_a\nabla^a\phi + \frac{1}{6}Scal_g\phi = \Omega^{-3}\left(\hat{\nabla}_a\hat{\nabla}^a\left(\Omega^{-1}\phi\right) + \frac{1}{6}Scal_{\hat{g}}\left(\Omega^{-1}\phi\right)\right)$$

where $Scal_g$ and $Scal_{\hat{g}}$ are the scalar curvatures associated with g and $\hat{g}$ respectively.

Assuming that we are working on a vacuum space-time, for which the scalar curvature vanishes, the equation becomes:

$$\nabla_a\nabla^a\phi = \Omega^{-3}\left(\hat{\nabla}_a\hat{\nabla}^a\left(\Omega^{-1}\phi\right) + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\left(\Omega^{-1}\phi\right)\right) \tag{1.1}$$

We obtain in particular the useful formula:

$$\Omega^3\nabla_a\nabla^a\Omega = \frac{1}{6}\mathrm{Scal}_{\hat{g}}. \tag{1.2}$$

Let us now consider the Cauchy problem on the physical spacetime M:

$$\left\{\begin{array}{l} \Box\phi + b\phi^3 = 0 \\ \phi\big|_{\Sigma_0} = \theta \in C_0^\infty(\Sigma_0) \\ \hat{T}^a\nabla_a\phi\big|_{\Sigma_0} = \xi \in C_0^\infty(\Sigma_0) \end{array}\right. . \tag{1.3}$$

Using this transformation, this Cauchy problem is transformed into a Cauchy problem on the unphysical spacetime $\hat{M}$ as followed:

Lemma 1.8. *The function ϕ is a solution of problem 1.3 if, and only if, the function*

$$\psi = \Omega^{-1}\phi$$

is solution of the problem on Σ_0:

$$\left\{\begin{array}{l} \hat{\Box}\psi + \frac{1}{6}Scal_{\hat{g}}\psi + b\psi^3 = 0 \\ \psi\big|_{\Sigma_0} = \Omega^{-1}\theta \in C_0^\infty(\Sigma_0) \\ \hat{T}^a\hat{\nabla}_a\psi\big|_{\Sigma_0} = \frac{1}{\Omega}\left(\xi - (\hat{T}^a\hat{\nabla}_a\Omega)\frac{\theta}{\Omega}\right) \in C_0^\infty(\Sigma_0) \end{array}\right. .$$

Remark 1.9. *1. Because of the finite speed propagation, since the data on the physical space-time are smooth with compact support on Σ_0, the data on the unphysical space-time are smooth with compact support in Σ_0.*

2. Another consequence of the finite speed propagation is that, because the data remain with compact support in Σ_0, we do not have to deal with the singularity in i^0.

3. Conversely, it is possible to start with a Cauchy problem on Σ_0 in the unphysical space-time and obtain a Cauchy problem on the physical space-time: starting with the Cauchy problem on $\hat{M}$:

$$\left\{\begin{array}{l} \hat{\square}\psi + \frac{1}{6}Scal_{\hat{g}}\psi + b\psi^3 = 0 \\ \psi\big|_{\Sigma_0} = \hat{\theta} \in C_0^\infty(\Sigma_0) \\ \hat{T}^a\hat{\nabla}_a\psi\big|_{\Sigma_0} = \hat{\xi} \in C_0^\infty(\Sigma_0) \end{array}\right. ,$$

then $\phi = \Omega\psi$ satisfies the Cauchy problem:

$$\left\{\begin{array}{l} \hat{\square}\phi + b\psi^3 = 0 \\ \psi\big|_{\Sigma_0} = \Omega\hat{\theta} \in C_0^\infty(\Sigma_0) \\ \hat{T}^a\nabla_a\psi\big|_{\Sigma_0} = \Omega\hat{\xi} + (\hat{T}^a\nabla_a\Omega)\hat{\theta} \in C_0^\infty(\Sigma_0) \end{array}\right. .$$

1.2.2 Function spaces

The purpose of this section is to give a precise description of the Sobolev spaces which are used in the present paper. Two problems are encountered in this section: the first one consists in adapting the definition of Sobolev spaces on a null hypersurface and the second is the difficulty coming from the singularity at i^0. This difficulty has two aspects: the necessity to adapt the definition of the Sobolev space to the singularity: this is solved using weighted Sobolev spaces on $\mathfrak{I}$. Another aspect of this singularity is encountered in section 3.1.3.

We recall the definition of a Sobolev space on a uniformly spacelike hypersurface Σ: for a smooth function u on Σ, consider the norm, when the integral exists:

$$||u||_p^2 = \int_\Sigma \sum_{k=0}^{p} ||D^k u||_h^2 \mathrm{d}\mu[h],$$

where h is the restriction of the metric $\hat{g}$ and D is the restriction of the connection $\hat{\nabla}$ to Σ.

Definition 1.10. *The completion of the space:*

$$\left\{u \in C^\infty(\Sigma)\big| \, ||u||_p < +\infty\right\}$$

in the norm $||\star||_p$is denoted by $H^p(\Sigma)$.
When Σ is is a compact spacelike hypersurface with boundary, the completion of the space of smooth functions with compact support in the interior of Σ in the norm $||\star||_p$is denoted by $H_0^p(\Sigma)$.

Remark 1.11. *It is known that, when working on a Riemannian closed manifold, the Sobolev spaces are independent of the choice of the metric. Nonetheless, this fact is not true any more when working with a weakly spacelike hypersurface, as we are about to see. Arbitrary choices are made for their definitions.*

Because of the degeneracy of the metric, it is not possible to define on $\mathscr{I}^+$ geometric quantities that only depend on the metric $\hat{g}$. Two solutions can be provided:

- lifting the metric from Σ_0 to $\mathscr{I}^+$;
- adding geometric information on $\mathscr{I}^+$ by using the uniformly timelike vector field $\hat{T}^a$.

Following [49, 55] and using Geroch-Held-Penrose formalism, $\mathscr{I}^+$ is endowed with a basis (l^a, n^a, e_3^a, e_4^a) such that:

- l^a and n^a are two future directed null vectors; n^a is tangent to $\mathscr{I}^+$; they satisfy:
$$l^a + n^a = \hat{T}^a;$$
in the neighborhood of i^0, they are chosen to be:
$$l^a = -2\partial_R \text{ and } n^a = u^2\partial_u$$
- the set $\{l^a, n^a\}$ is completed by two vectors (e_3^a, e_4^a) orthogonal to $\{l^a, n^a\}$, orthogonal to each other and normalized.

$\mathscr{I}^+$ is then endowed with the volume form $i^\star_{\mathscr{I}^+}(l^a \lrcorner \mu[\hat{g}])$. The following norm is defined on $\mathscr{I}^+$, for u a smooth function with compact support which does not contain i^0 or i^+:

$$||u||^2_{H^1(\mathscr{I}^+)} = \int_{\mathscr{I}^+} \left(\frac{\left(n^a \hat{\nabla} u\right)^2}{\hat{g}_{cd}\hat{T}^c\hat{T}^d} + \left|\hat{\nabla}_{\mathbb{S}^2} u\right|^2 + u^2 \right) i^\star_{\mathscr{I}^+}(l^a \lrcorner \mu[\hat{g}])$$

where $\left|\hat{\nabla}_{\mathbb{S}^2} u\right|$ stands for the derivatives with respect to (e_3^a, e_4^a).

Following chapter 5.4.3 of Friedlander's book ([39]), the Sobolev space H^1 on $\mathscr{I}^+$ is finally defined:

Definition 1.12. *Let $\tilde{M}$ be an extension of $\hat{M}$ behind i^+ and consider the function space $\mathcal{D}(\mathscr{I}^+)$ on $\mathscr{I}^+$ obtained as the trace of smooth functions with compact support in $\tilde{M}$ which does not contain i^0. The weighted Sobolev space $H^1(\mathscr{I}^+)$ is defined as the completion of the space $\mathcal{D}(\mathscr{I}^+)$ in the norm $|| \star ||_{H^1(\mathscr{I}^+)}$.*

Since the volume form is written on the Schwarzschildean part of $\hat{M}$ as, using polar coordinates:
$$\mu[\hat{g}] = \sin(\theta)\mathrm{d}u \wedge \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\psi,$$
this norm is written in the Schwarzschild part of the manifold:

$$||\phi||^2_{H^1(\mathscr{I}^+)} = 2\int_{\mathscr{I}^+} \left(\frac{1}{4}u^2(\partial_u\phi)^2 + |\hat{\nabla}_{\mathbb{S}^2}\phi|^2 + \phi^2 \right) \mathrm{d}u\mathrm{d}\omega_{\mathbb{S}^2}$$

since, on $\mathscr{I}^+$,
$$n^a = u^2\partial_u \text{ and } \hat{g}_{cd}\hat{T}^c\hat{T}^d = 4u^2$$

The metric at i^+ is obtained as the restriction of a smooth metric of an extension of $\hat{M}$ beyond $\mathcal{I}^+$. As a consequence, a trace theorem could give another way, more intrinsic, to define $H^1(\mathcal{I}^+ \cap O)$ where O is a bounded open set around i^+. Another way to obtain the fact that the point i^+ does not matter in the definition of the Sobolev space over $\mathcal{I}^+$ is to notice the following property:

Proposition 1.13. *The set of smooth functions with compact support on $\mathcal{I}^+$, whose support does not contain i^+ is dense in $H^1(\mathcal{I}^+)$.*

Proof. The method of the proof relies on the construction of an identification between $H_0^1(\Sigma_0)$ and $H^1(\mathcal{I}^+)$. This identification is brought by Hörmander in [53] and is obtained as follows.

Let t be a smooth time function in the future of Σ_0 in $\hat{M}$. This time function gives rise to a local coordinate system, where t is the time coordinate. We denote by ∂_t the vector field associated with this coordinate. The flow associated with this vector field is denoted Ψ_t.

For x in Σ_0, let $\phi(x)$ be the time at which the curve $\Psi_t(x)$ hits $\mathcal{I}^+$ and consider the application defined by:

$$\xi \in C_0^\infty(\Sigma_0) \longmapsto \left(y \in \mathcal{I}^+ \mapsto \xi\left(\Psi_{-\phi(y)}(y)\right)\right)$$

This application has value in $C_0^\infty(\mathcal{I}^+)$ since the future of a compact set in Σ_0 has compact support in $\mathcal{I}^+$. Furthermore, this application can easily be inverted:

$$\xi \in C_0^\infty(\mathcal{I}^+) \longmapsto \left(x \in \mathcal{I}^+ \mapsto \xi\left(\Psi_{-\phi(x)}(x)\right)\right) \in C_0^\infty(\Sigma_0)$$

and can consequently be used to define on $\mathcal{I}^+$ a Sobolev space by pushing forward the H^1-norm on Σ_0. The Sobolev spaces which are obtained are then equivalent on $H^1(\mathcal{I}^+)$ since the norm are equivalent on any compact set of $\mathcal{I}^+$.

To prove the theorem, it is then sufficient to prove that the set of smooth functions with compact support in Σ_0 wich does not contain the preimage of i^+ by the flow associated wit the time function t is dense in $H^1(\Sigma_0)$. Since Σ_0 has no topology, it is sufficient to establish the following lemma:

Lemma 1.14. *Let us consider the set of smooth functions defined in $\overline{B(0,1)} \subset \mathbb{R}^3$ with support which does not contain 0. Then this set is dense in $H^1(B(0,1))$.*

Proof. It is sufficient to prove that the constant function 1 can be approximated by a sequence of smooth function whose compact support does not contain 0.

Let f be the function defined on $\mathbb{R}^+$ by:

- f is a smooth function on $\mathbb{R}^+$ with value in $[0,1]$;
- $f = 1$ in $[\frac{1}{2}, +\infty)$;
- f vanishes in $[0, \frac{1}{3}]$.

Let us consider the sequence of smooth spherically symmetric functions defined by:

$$\forall n \in \mathbb{N}, \forall x \in B(0,1), \psi_n(x) = f(n||x||).$$

They satisfies, for all n in $\mathbb{N}$:

- $\psi_n = 1$ in $B(0,1)\backslash B(0,\frac{1}{2n})$;
- ψ_n vanishes in $B(0,\frac{1}{3n})$;
- ψ_n is a smooth function on $B(0,1)$ with value in $[0,1]$ since it vanishes in a neighborhood of zero.

Finally, the difference $(1-\psi_n)_n$ converges towards 0 in H^1-norm:

$$\begin{aligned} ||1-\psi_n||^2_{H^1} &= \int_{B(0,1)} \left((1-f(nr))^2 + n^2(f'(nr))^2\right) r^2 \mathrm{d}r\mathrm{d}\omega_{\mathcal{S}^2} \\ &= \frac{4}{3}\pi\left(\int_0^1 (1-f(nr))^2 \mathrm{d}r + n^2 \int_{\frac{1}{3n}}^{\frac{1}{2n}} (f'(nr))^2 r^2 \mathrm{d}r\right) \\ &\leq \frac{4}{3}\pi\left(\int_0^1 (1-f(nr))^2 \mathrm{d}r + \frac{\sup_{\mathbb{R}}((f')^2)}{n}\right). \end{aligned}$$

The remaining integral converges towards 0 by Lebesgue theorem. As a consequence, the sequence $(\psi_n)_n$ converges towards the constant 1 in $H^1(B(0,1))$.

Finally, let f be a function $H^1(B(0,1))$ (or in $H^1_0(B(0,1))$). Cauchy-Schwarz inequality gives, for all n in $\mathbb{N}$:

$$\begin{aligned} ||f(1-\psi_n)||^2_{H^1} &= ||f(1-\psi_n)||^2_{L^2} + \left|\left|f|\nabla(1-\psi_n)|\right|\right|^2_{L^2} + \left|\left||\nabla f|(1-\psi_n)\right|\right|^2_{L^2} \\ &\leq 2||f||^2_{H^1}||1-\psi_n||^2_{H^1}. \end{aligned}$$

$(f\psi_n)_n$ is then a sequence of functions in $H^1(B(0,1))$ whose support does not contain 0 which converges in H^1 towards f. ♦

Using this lemma in the neighborhood of the preimage of i^+ immediatly gives the result. ♦

1.3 Cauchy problem

A well-posedness theorem in our framework is now stated. It is based on a result of Choquet-Bruhat and Cagnac in [11] (and see also [14], appendix III for a survey on the wave equation, and appendix III chapter 5 for our problem).

The geometric framework for this well-posedness theorem is the following (definition 11.8 in [14]):

Definition 1.15. *A space-time (M,g) is said to be regularly sliced if there exists a smooth 3-manifold Σ endowed with coordinates (x^i) and an interval I of $\mathbb{R}$ such that M is $I\times\Sigma$ and the metric g can be written:*

$$g = N^2 dt^2 - g_{ij}(dx^i + \beta^i dt).$$

and its coefficients satisfy:

1. *the lapse function N is bounded above and below by two positives constants:*

$$\exists(c,C), 0 < c \leq N \leq C;$$

2. *for t in I, the 3-dimensional Riemannian manifolds* $(\{t\} \times M, g_{t,ij} = g_{ij}\big|_{\{t\}\times M})$ *are complete and the metrics* g_t *are bounded below by a metric h i.e.:*

$$\forall V \in T\Sigma, h_{ij}V^iV^j \leq g_{t,ij}V^iV^j;$$

3. *and, finally, the norm for the metric* g_t *of the vector* β *is uniformly bounded on* M*.*

Remark 1.16. *1. This hypothesis implies that the space-time is globally hyperbolic (theorem 11.10 in [14]).*

2. *The asymptotically simple space-time and its compacification which we are working with do not satisfy this property.*

The following theorem, obtained by Choquet-Bruhat and Cagnac in [11] , gives existence and uniqueness of solutions to the Cauchy problem for a cubic wave equation:

Theorem 1.17 (Cauchy problem for a nonlinear wave equation)**.** *Let us consider the Cauchy problem on the regularly sliced manifold* $(M = \mathbb{R} \times \Sigma, g)$*:*

$$\begin{cases} \hat{\Box}\phi + \frac{1}{6}Scal_{\hat{g}}\phi + b\phi^3 = 0 \\ \phi\big|_{\{0\}\times\Sigma} = \theta \in H^1(\Sigma) \\ \partial_t\phi\big|_{\{0\}\times\Sigma} = \tilde{\theta} \in L^2(\Sigma) \end{cases}$$

Then this problem admits a unique global solution on M *in* $C^0(\mathbb{R}, H^1(\Sigma)) \cap C^1(\mathbb{R}, L^2(\Sigma))$*.*

As already noted (see remark 1.16), this theorem cannot of course be applied directly to the compactification of M because of the geometry in the Schwarzschild part. This problem can be solved using an extension of $\hat{M}$ constructed as follows:

1. Let θ and ξ be two function in $H^1(\Sigma_0)$ and $L^2(\Sigma_0)$ with compact support in the interior of Σ_0. Let K be a compact subset of Σ_0 containing the support of θ and ξ.

2. Let $\hat{t}$ be a time function on $\hat{M}$ such that Σ_0 is given by $\{\hat{t} = 0\}$; the associated foliation is denoted by $(\Sigma_{\hat{t}})$ for $\hat{t} \in [0, \hat{T}]$; we assume that the gradient of this time function is uniformly timelike for the metric $\hat{g}$.

3. The manifold $(J^+(K), g)$ is a 4-dimensional Lorentzian manifold with boundary. This boundary is constituted of K, the light cone from K, $C^+(K)$, and the part of $\mathscr{I} \cup \{i^+\}$ in the future of K. Since $(\hat{M}, g)$ is extendible smoothly in the neighborhood of i^+, there exists a smooth extension of $(J^+(K), g)$ into a 4-dimensional Lorentzian manifold $(\tilde{M}, \tilde{g})$, depending on the support of K such that the manifold $\tilde{M}$ is diffeomorphic to $[0, \hat{T}] \times \tilde{\Sigma}$, where $\tilde{\Sigma}$ is topologically equivalent to $\mathcal{S}^3$ and such that the foliation $(\Sigma_{\hat{t}})$ is extended into the uniformly spacelike foliation $(\{t\} \times \tilde{\Sigma})$.

Since $\tilde{M}$ is compact, conditions 1 ,2 and 3 of definition 1.15 are clearly satisfied. As a consequence, using theorem 1.17, we obtain the well-posedness in $C^0(\mathbb{R}, H^1(\Sigma_0))$ of the Cauchy problem for

$$\begin{cases} \hat{\Box}\phi + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\phi + b\phi^3 = 0 \\ \phi\big|_{\{0\}\times\Sigma} = \theta \in H^1(\Sigma) \text{ with compact support in } K \\ \partial_t\phi\big|_{\{0\}\times\Sigma} = \tilde{\theta} \in L^2(\Sigma) \text{ with compact support in } K \end{cases}$$

As a consequence, we obtain by restriction to $J^+(K)$ and on $\hat{M}$ well posedness of the Cauchy problem for data in $H^1_0(\Sigma_0) \times L^2(\Sigma_0)$.

Remark 1.18. *1. The extension of the solution of the wave equation to a cylinder was also used by Mason-Nicolas in [63] (proposition 6.1) to obtain energy estimates.*

2. The space $C^0(\mathbb{R}, H^1(\Sigma_0)) \cap C^1(\mathbb{R}, L^2(\Sigma_0)) \subset L^\infty(\mathbb{R}, L^2(\Sigma_0))$ is the space in which the Cauchy problem is well-posed. $\hat{M}$ is nonetheless not diffeomorphic to a product $\mathbb{R} \times \Sigma$. If $\hat{M}$ is extended in the same way, it is then possible to set a well-posedness theorem in this space.

2 A priori estimates

The purpose of this section is to establish a priori estimates for solutions of the wave equation, in the sense that it is possible to control the energy on $\mathcal{I}^+$ by the energy on Σ_0 and reciprocally. These a priori estimates will be used in the next section to establish the continuity of the conformal wave operator, its domain of definition and the existence of trace operators.

Let us consider in this section a smooth solution with compactly supported data of the problem:

$$\hat{\Box}\phi + \tfrac{1}{6}\mathrm{Scal}_{\hat{g}}\phi + b\phi^3 \quad = \quad 0 \tag{2.1}$$

and the associated stress-energy tensor:

$$T_{ab} = \hat{\nabla}_a\phi\hat{\nabla}_b\phi + \hat{g}_{ab}\left(-\frac{1}{2}\hat{\nabla}_c\phi\hat{\nabla}^c\phi + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right).$$

The contraction of this tensor with $\hat{T}^a$, $\hat{T}^a T_{ab}$, is called "energy 3-form"; it satisfies an "approximate conservation law":

Lemma 2.1. *The derivative of the energy 3-form satisfies:*

$$\hat{\nabla}^a\left(\hat{T}^b T_{ab}\right) \quad = \quad \left(\hat{\nabla}^{(a}\hat{T}^{b)}\right)T_{ab} + \left(1 - \frac{1}{6}Scal_{\hat{g}}\right)\phi\hat{T}^a\hat{\nabla}_a\phi + \hat{T}^a\hat{\nabla}_a b\frac{\phi^4}{4}.$$

This derivative will be designated as "the error term" since it arises in the volume term when applying Stokes theorem.

A quantity which is equivalent to the integral:

$$\int_S i^\star(\star T^a T_{ab})$$

on the hypersurface S is called an energy and is denoted by $E(S)$. A global $E(S)$ is piecewise defined in the future of Σ_0 in propositions 2.5, 2.16 and 2.21. The purpose of this multiple definitions is to simplify the comparison with other quantities.

2.1 Estimates in the neighborhood of i^0

The purpose of this section is to obtain a priori estimates for the energy associated with the energy 3-form on $\mathcal{I}^+$ and Σ_0, using the fact that the geometry is known almost completely.

The estimates which are obtained in this section are close to the ones obtained for the linear wave equation by Mason-Nicolas in [63]. These estimates are based on two main tools:

- an explicit control of the decay of the physical metric in the neighborhood of i^0

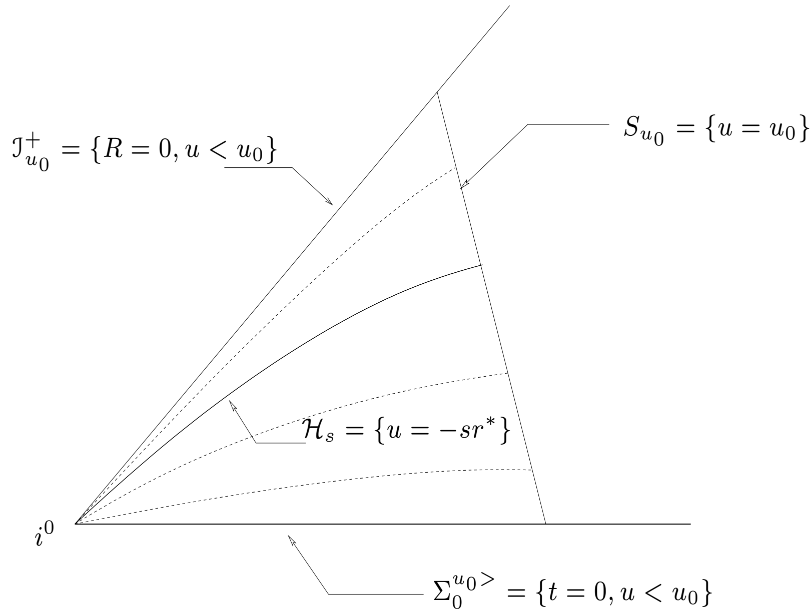


Figure 2.1: Neighborhood of i^0

- and the use of Gronwall lemma.

We define, in $\Omega^+_{u_0} = \{t > 0, u < u_0\}$, the following hypersurfaces, for u_0 given in $\mathbb{R}$:

- $S_{u_0} = \{u = u_0\}$, a null hypersurface transverse to $\mathcal{I}^+$;
- $\Sigma_0^{u_0>} = \Sigma_0 \cap \{u_0 > u\}$, the part of the initial data surface Σ_0 beyond S_{u_0};
- $\mathcal{I}^+_{u_0} = \Omega^+_{u_0} \cap \mathcal{I}^+$, the part of $\mathcal{I}^+$ beyond S_{u_0};
- $\mathcal{H}_s = \Omega^+_{u_0} \cap \{u = -sr^*\}$, for s in $[0, 1]$, a foliation of $\Omega^+_{u_0}$ by spacelike hypersurfaces accumulating on $\mathcal{I}$.

The volume form associated with $\hat{g}$ in the coordinates $(R, u, \omega_{\mathbb{S}^2})$ is then:

$$\mu[\hat{g}] = \mathrm{d}u \wedge \mathrm{d}R \wedge \mathrm{d}^2\omega_{\mathbb{S}^2}. \tag{2.2}$$

Finally, we consider the approximate conformal Killing vector field $\hat{T}^a$:

$$\hat{T}^a = u^2\partial_u - 2(1 + uR)\partial_R.$$

Remark 2.2. *1. This vector field is timelike for the unphysical metric and, as such, is transverse to $\mathcal{I}$. More precisely, it is uniformly timelike in a neighborhood of i^0 (see remark 1.6 for the choice of the neighborhood).*

2. Its expression is derived from the so-called Morawetz vector field in the Minkowski space and was previously used to obtain pointwise estimates in the flat case.

The strategy of the proof in this section is the following:

1. writing an explicit description of the hypersurfaces S_{u_0} and $\mathcal{H}_s^{u_0}$;

2. proving energy equalities in both ways for $\int \star\hat{T}^a T_{ab}$ using the Stokes theorem between the hypersurfaces $\Sigma^+_{u_0}$ and $\mathcal{H}^{u_0}_s$;

3. determining an energy $E(\mathcal{H}^{u_0}_s)$ equivalent to $\int \star\hat{T}^a T_{ab}$ from the decay of the metric g;

4. obtaining an integral inequality for $E(\mathcal{H}^{u_0}_s)$ to apply the Gronwall lemma;

5. starting from point 2, doing the same work from a Stokes theorem applied between $\mathcal{I}^+_{u_0}$ and $\mathcal{H}^{u_0}_s$.

2.1.1 Geometric description

This section is devoted to the description of the energy associated with the nonlinear wave equation in the neighborhood of i^0.

Proposition 2.3. *The energy 3-form, written in the coordinates (R, u, θ, ψ), is given by:*

$$\begin{aligned}\star\hat{T}^a T_{ab} = &\Big[u^2(\partial_u\phi)^2 + R^2(1-2mR)\left(u^2\partial_R\phi\partial_u\phi - (1+uR)(\partial_R\phi)^2\right)\\ &+\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|^2 + \frac{\phi^2}{4} + b\frac{\phi^4}{4}\right)\Big]\sin(\theta)\,du\wedge d\theta\wedge d\psi\\ +&\left[\frac{1}{2}\left((2+uR)^2 - 2mR^3u^2\right)\left(\partial_R\phi^2\right) + u^2\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\right]\sin(\theta)\,dR\wedge d\theta\wedge d\psi\\ &+\sin(\theta)\left[u^2\partial_u\phi - 2(1+uR)\partial_R\phi\right]\left(-\partial_\theta\phi du\wedge dR\wedge d\psi + \partial_\psi\phi du\wedge dR\wedge d\theta\right)\end{aligned}$$

The restriction of the energy 3-form can be written:

- *to $\mathcal{H}_s$:*

$$\begin{aligned}i^\star_{\mathcal{H}_s}(\star\hat{T}^a T_{ab}) = &\Big(u^2(\partial_u\phi)^2 + R^2(1-2mR)u^2\partial_R\phi\partial_u\phi\\ &+R^2(1-2mR)\left(\frac{(2+uR)^2}{2s} - \frac{mu^2R^3}{s} - (1+uR)\right)(\partial_R\phi^2)\\ &+\left(\frac{u^2R^2(1-2mR)}{s} + 2(1+uR)\right)\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi| + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\Big)\sin(\theta)\,du\wedge d\theta\wedge d\psi;\end{aligned}$$

- *to S_u:*

$$\begin{aligned}i^\star_{S_u}(\star\hat{T}^a T_{ab}) = &\Big(\frac{1}{2}\left((2+uR)^2 - 2mR^3u^2\right)\left(\partial_R\phi^2\right)\\ &+u^2\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\Big)\sin(\theta)\,dR\wedge d\theta\wedge d\psi;\end{aligned}$$

- *to $\mathcal{I}^+_{u_0}$*

$$i^\star_{\mathcal{I}^+}(\star\hat{T}^a T_{ab}) = \left(u^2(\partial_u\phi)^2 + \|\nabla_{\mathbb{S}^2}\phi| + \phi^2 + b\frac{\phi^4}{2}\right)\sin(\theta)\,du\wedge d\theta\wedge d\psi.$$

Proof. For the calculation to come, let us denote by A, the quantity:

$$A = -\frac{1}{2}\hat{\nabla}_c\phi\hat{\nabla}^c\phi + \frac{\phi^2}{2} + b\frac{\phi^4}{4}$$

We calculate the general form of $\star T^a T_{ab}$:

$$\begin{aligned}\star\hat{T}^a T_{ab} &= \star(u^2\partial_u^a - 2(1+uR)\partial_R^a)T_{ab} \\ &= (u^2\partial_u\phi - 2(1+uR)\partial_R\phi)\star\hat{\nabla}_b\phi + u^2 A\partial_u^a \lrcorner\mu[\hat{g}] - 2A(1+uR)\partial_R\lrcorner\mu[\hat{g}].\end{aligned}$$

Using the expression of the volume form in the coordinate system (equation (2.2)), we obtain:

$$\begin{array}{rclrcl}\partial_u\lrcorner\mu[\hat{g}] &=& \mathrm{d}R\wedge\mathrm{d}^2\omega_{\mathbb{S}^2}, & \partial_\theta\lrcorner\mu[\hat{g}] &=& \sin(\theta)\mathrm{d}R\wedge\mathrm{d}u\wedge\mathrm{d}\theta, \\ \partial_R\lrcorner\mu[\hat{g}] &=& -\mathrm{d}u\wedge\mathrm{d}^2\omega_{\mathbb{S}^2}, & \partial_\phi\lrcorner\mu[\hat{g}] &=& -\sin(\theta)\mathrm{d}R\wedge\mathrm{d}u\wedge\mathrm{d}\phi.\end{array}$$

$\hat{\nabla}\phi$ is written in the coordinates (R, u, ω_{S^2}) as follows:

$$\hat{\nabla}\phi = -\partial_R\phi\partial_u - (\partial_u\phi + R^2(1-2mR)\partial_R\phi)\partial_R - \hat{\nabla}_{\mathbb{S}^2}\phi;$$

its norm is then:

$$\begin{aligned}\hat{\nabla}_c\phi\hat{\nabla}^c\phi &= (-\partial_R\phi)^2 g(\partial_u,\partial_u) + 2\partial_R\phi(\partial_u\phi + R^2(1-2mR)\partial_R\phi))g(\partial_u,\partial_R) - |\hat{\nabla}_{\mathbb{S}^2}\phi|^2 \\ &= -R^2(1-2mR)(\partial_R\phi)^2 - 2\partial_R\phi\partial_u\phi - |\hat{\nabla}_{\mathbb{S}^2}\phi|^2.\end{aligned}$$

The Hodge dual of $\nabla_a\phi$ is calculated by splitting $\hat{\nabla}_a u$ over $(\mathrm{d}u, \mathrm{d}R, \mathrm{d}\omega_{S^2})$:

$$\begin{array}{rcl}\star\hat{\nabla}_b\phi &=& \star(\partial_u\phi\mathrm{d}u + \partial_R\phi\mathrm{d}R + \partial_\theta\phi\mathrm{d}\theta + \partial_\psi\phi\mathrm{d}\psi) \\ &=& \partial_u\phi\mathrm{d}R\wedge\mathrm{d}\omega_{\mathbb{S}^2} - \partial_R\phi\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\end{array}$$

The gradient of each coordinates is calculated:

$$\begin{array}{rclrcl}\hat{\nabla}^b u &=& \hat{g}^{ab}\hat{\nabla}_a u = -\partial_R & \hat{\nabla}^b\theta &=& -\sin(\theta)\partial_\theta \\ \hat{\nabla}^b R &=& -\partial_u - R^2(1-2mR)\partial_R & \hat{\nabla}^b\psi &=& -\partial_\psi,\end{array}$$

and, as a consequence,

$$\begin{array}{rclrcl}\star\mathrm{d}u &=& \mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2} & \star\mathrm{d}\theta &=& -\sin\theta\mathrm{d}u\wedge\mathrm{d}R\wedge\mathrm{d}\psi \\ \star\mathrm{d}R &=& -\mathrm{d}R\wedge\mathrm{d}\omega_{\mathbb{S}^2} + R^2(1-2mR)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2} & \star\mathrm{d}\psi &=& \mathrm{d}u\wedge\mathrm{d}R\wedge\mathrm{d}\theta\end{array},$$

so that $\star\nabla_a\phi$ is:

$$\begin{array}{rcl}\star\nabla_a\phi &=& \partial_u\phi\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2} + \partial_R\phi\left(-\mathrm{d}R\wedge\mathrm{d}\omega_{\mathbb{S}^2} + R^2(1-2mR)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\right) \\ && -\partial_\theta\phi\sin\theta\mathrm{d}u\wedge\mathrm{d}R\wedge\mathrm{d}\psi + \partial_\psi\mathrm{d}u\wedge\mathrm{d}R\wedge\mathrm{d}\theta \\ &=& \left(\partial_u\phi + R^2(1-2mR)\partial_R\phi\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2} - \partial_R\phi\mathrm{d}R\wedge\mathrm{d}\omega_{\mathbb{S}^2} \\ && -\partial_\theta\phi\sin\theta\mathrm{d}u\wedge\mathrm{d}R\wedge\mathrm{d}\psi + \partial_\psi\mathrm{d}u\wedge\mathrm{d}R\wedge\mathrm{d}\theta\end{array}$$

The energy 3-form is then:

$$\begin{aligned}\star\hat{T}^a T_{ab} = \Big[&u^2(\partial_u\phi)^2 + R^2(1-2mR)\left(u^2\partial_R\phi\partial_u\phi - (1+uR)(\partial_R\phi\right)^2 \\ &+2(1+uR)\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\Big]\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2} \\ &+\left[\frac{1}{2}\left((2+uR)^2 - 2mR^3u^2\right)\left(\partial_R\phi^2\right) + u^2\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\right]\mathrm{d}R\wedge\mathrm{d}\omega_{\mathbb{S}^2} \\ &+\sin(\theta)\left[u^2\partial_u\phi - 2(1+uR)\partial_R\phi\right](-\partial_\theta\phi\mathrm{d}u\wedge\mathrm{d}R\wedge\mathrm{d}\psi + \partial_\psi\phi\mathrm{d}u\wedge\mathrm{d}R\wedge\mathrm{d}\theta)\end{aligned}$$

Remark 2.4. *To calculate the restrictions of $\star\hat{T}^a T_{ab}$ to the hypersurfaces $\mathcal{H}_s$, S_u and $\mathcal{I}^+$, it is necessary to give the restrictions of each of the differentials of the coordinates. Nonetheless, since ∂_θ and ∂_ϕ are tangent to $\mathcal{H}_s$, S_u and $\mathcal{I}^+$, the only remaining 3-forms to consider when restricting to these hypersurfaces are $du\wedge d\omega_{\mathbb{S}^2}$ and $dR\wedge d\omega_{\mathbb{S}^2}$. This means that only du and dR should be taken care of.*

Noticing that

$$\frac{\mathrm{d}r^*}{\mathrm{d}R} = \frac{-1}{R^2(1-2mR)},$$

we get, on $\mathcal{H}_s$, defined in $\Omega^+_{u_0}$ by $u = -sr^*$:

$$\mathrm{d}R|_{\mathcal{H}_s} = \frac{R^2(1-2mR)}{s}\,\mathrm{d}u|_{\mathcal{H}_s} = \frac{r^*R^2(1-2mR)}{|u|}\,\mathrm{d}u|_{\mathcal{H}_s}\,.$$

The restriction to $\mathcal{H}_s$ of the energy 3-form is then:

$$\begin{aligned} i^\star_{\mathcal{H}_s}(\star\hat{T}^a T_{ab}) = \Big(\Big[&u^2(\partial_u\phi)^2 + R^2(1-2mR)\left(u^2\partial_R\phi\partial_u\phi - (1+uR)(\partial_R\phi)^2\right.\\ &+2(1+uR)\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|^2+\frac{\phi^2}{2}+b\frac{\phi^4}{4}\right)\Big]\\ +\frac{R^2(1-2mR)}{s}&\left[\frac{1}{2}\left((2+uR)^2-2mR^3u^2\right)(\partial_R\phi^2)+u^2\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|^2+\frac{\phi^2}{2}+b\frac{\phi^4}{4}\right)\right]\\ &\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\end{aligned}$$

$$\begin{aligned} i^\star_{\mathcal{H}_s}(\star\hat{T}^a T_{ab}) &= \big(u^2(\partial_u\phi)^2 + R^2(1-2mR)u^2\partial_R\phi\partial_u\phi\\ &+R^2(1-2mR)\left(\frac{(2+uR)^2}{2s}-\frac{mu^2R^3}{s}-(1+uR)\right)(\partial_R\phi^2)\\ &+\left(\frac{u^2R^2(1-2mR)}{s}+2(1+uR)\right)\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|+\frac{\phi^2}{2}+b\frac{\phi^4}{4}\right)\Big)\,\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}.\end{aligned}$$

$\mathcal{I}^+$ is defined by $R=0$; so the restriction to $\mathcal{I}^+$ of the energy 3-form is:

$$i^\star_{\mathcal{I}^+}(\star\hat{T}^a T_{ab}) = \left(u^2(\partial_u\phi)^2 + \left(|\nabla_{\mathbb{S}^2}\phi|+\phi^2+b\frac{\phi^4}{2}\right)\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}.$$

Finally, for S_u which is defined by $\{u = constant\}$, we obtain:

$$i^\star_{S_u}(\star\hat{T}^a T_{ab}) = \left(\frac{1}{2}\left((2+uR)^2-2mR^3u^2\right)(\partial_R\phi^2)+u^2\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|^2+\frac{\phi^2}{2}+b\frac{\phi^4}{4}\right)\right)\mathrm{d}R\wedge\mathrm{d}\omega_{\mathbb{S}^2}.\blacklozenge$$

Proposition 2.5. *There exists u_0, such that the following energy estimates holds on $\mathcal{H}_s$ in $\Omega^+_{u_0}$:*

$$\int_{\mathcal{H}_s} i^\star_{\mathcal{H}_s}\left(\star\hat{T}^a T_{ab}\right) \approx \int_{\mathcal{H}_s}\left(u^2(\partial_u\phi)^2+\frac{R}{|u|}(\partial_R\phi)^2+|\nabla_{\mathbb{S}^2}\phi|^2+\frac{\phi^2}{2}+b\frac{\phi^4}{4}\right)du\wedge d\omega_{\mathbb{S}^2}$$

Proof. Let us consider the expression of $i^\star_{\mathcal{H}_s}(\star\hat{T}^a T_{ab})$:

$$i^\star_{\mathcal{H}_s}(\star\hat{T}^a T_{ab}) = \Big(u^2(\partial_u\phi)^2 \tag{2.3}$$

$$+R^2u^2(1-2mR)\partial_R\phi\partial_u\phi \tag{2.4}$$

$$+R^2(1-2mR)\left(\frac{(2+uR)^2}{2s}-\frac{mu^2R^3}{s}-(1+uR)\right)(\partial_R\phi^2) \tag{2.5}$$

$$+\left(\frac{u^2R^2(1-2mR)}{s}+2(1+uR)\right)\left(\frac{1}{2}|\nabla_{\mathbb{S}^2}\phi|+\frac{\phi^2}{2}+b\frac{\phi^4}{4}\right)\Big)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2} \tag{2.6}$$

Each of these terms is estimated separately using lemma 1.5 and the obtained estimates are summed. Let ϵ be a given positive number and let u_0 be the non positive constant associated to ϵ via lemma 1.5; ϵ will be chosen during the proof.

Nothing needs to be done for (2.3).

For (2.6), since u is non-positive and $s = -\frac{u}{r^*} = \frac{|u|}{r^*}$, we have, on one side:

$$\begin{aligned}\left(\frac{u^2R^2(1-2mR)}{s} + 2(1-|u|R)\right) &= (Rr^*)(R|u|)(1-2mR) + 2(1-|u|R)\\ &\leq (Rr^*)(R|u|)(1-2mR) + 2\\ &\leq (1-\epsilon)(1+\epsilon) + 2\end{aligned}$$

and, on the other side:

$$\begin{aligned}\left(\frac{u^2R^2(1-2mR)}{s} + 2(1-|u|R)\right) &= (Rr^*)(1-2mR)(R|u|) + 2(1-|u|R)\\ &\geq 1\cdot(1-\epsilon)\cdot(R|u|) + 2(1-|u|R)\\ &\geq 2-(1+\epsilon)(R|u|)\\ &\geq 2-(1+\epsilon)(1+\epsilon)\\ &\geq 1-2\epsilon-\epsilon^2. \qquad (2.7)\end{aligned}$$

ϵ is chosen such as (2.7) is positive.

For (2.5), the proof slightly more complicated. We have, since u is non-positive and $s = -\frac{u}{r^*} = \frac{|u|}{r^*}$, on one hand:

$$\begin{aligned}&R^2(1-2mR)\left(\frac{(2+uR)^2}{2s} - \frac{mu^2R^3}{s} - (1+uR)\right)(\partial_R\phi)^2\\ &= R^2(1-2mR)\left(\frac{r^*(2-|u|R)^2}{2|u|} - (mR)(R|u|)(Rr^*) - (1+uR)\right)(\partial_R\phi)^2\\ &= \left(\frac{R}{|u|}(\partial_R\phi)^2\right)(1-2mR)(Rr^*)\left(\frac{(2-|u|R)^2}{2} - (mR)(R|u|)^2 - \frac{|u|}{r^*} + \frac{|u|}{r^*}R|u|\right) \qquad (2.8)\\ &\leq \left(\frac{R}{|u|}(\partial_R\phi)^2\right)\cdot 1\cdot(1+\epsilon)\left(\frac{(3+\epsilon^2)^2}{2} + 1\cdot(1+\epsilon)\right).\end{aligned}$$

Starting from equation (2.8), the lower bound for (2.5) is obtained as follows, setting $X = |u|R$:

$$\begin{aligned}&\left(\frac{R}{|u|}(\partial_R\phi)^2\right)(1-2mR)(Rr^*)\left(\frac{(2-|u|R)^2}{2} - (mR)(R|u|)^2 - \frac{|u|}{r^*} + \frac{|u|}{r^*}R|u|\right)\\ &\geq \left(\frac{R}{|u|}(\partial_R\phi)^2\right)(1-\epsilon)\cdot\left(\frac{(2-X)^2}{2}(Rr^*) - (mR)(Rr^*)X^2 - X + X^2\right)\\ &\geq \left(\frac{R}{|u|}(\partial_R\phi)^2\right)\frac{(1-\epsilon)}{2}\left(4-6X+3X^2-\epsilon(1+\epsilon)X^2\right)\end{aligned}$$

The polynomial $4-6X+3X^2$ reaches its minimum for $X=1$ and equals 1 at $X=1$ so that:

$$\begin{aligned}&\left(\frac{R}{|u|}(\partial_R\phi)^2\right)(1-2mR)(Rr^*)\left(\frac{(2-|u|R)^2}{2} - (mR)(R|u|)^2 - \frac{|u|}{r^*} + \frac{|u|}{r^*}R|u|\right)\\ &\geq \left(\frac{R}{|u|}(\partial_R\phi)^2\right)\frac{(1-\epsilon)}{2}\left(1-\epsilon(1+\epsilon)^3\right)\end{aligned}$$

To deal with (2.4), we write:

$$|R^2u^2(1-2mR)\partial_R\phi\partial_u\phi| = (1-2mR)\left(R^2|u|\sqrt{\frac{2}{3}}\partial_R\phi\right)\left(\sqrt{\frac{3}{2}}u\partial_u\phi\right),$$

so that:

$$\begin{aligned}|R^2u^2(1-2mR)\partial_R\phi\partial_u\phi| &\leq (1-2mR)\frac{1}{2}\left(\frac{3}{2}\left(u\partial_u\phi\right)^2+\frac{2}{3}(R^3|u|^3)\frac{R}{|u|}(\partial_R\phi)^2\right)\\ &\leq \frac{3}{4}\left(u\partial_u\phi\right)^2+\frac{1}{3}(1+\epsilon)^3\frac{R}{|u|}(\partial_R\phi)^2\end{aligned}$$

Finally, the following equivalence estimates hold:

$$c_\epsilon\int_{\mathcal{H}_s}\left(u^2(\partial_u\phi)^2+\frac{R}{|u|}(\partial_R\phi)^2+|\nabla_{\mathbb{S}^2}\phi|^2+\frac{\phi^2}{2}+b\frac{\phi^4}{4}\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\leq\int_{\mathcal{H}_s}i^\star_{\mathcal{H}_s}\left(\star\hat{T}^aT_{ab}\right)$$

and

$$\int_{\mathcal{H}_s}i^\star_{\mathcal{H}_s}\left(\star\hat{T}^aT_{ab}\right)\leq C_\epsilon\int_{\mathcal{H}_s}\left(u^2(\partial_u\phi)^2+\frac{R}{|u|}(\partial_R\phi)^2+|\nabla_{\mathbb{S}^2}\phi|^2+\frac{\phi^2}{2}+b\frac{\phi^4}{4}\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}$$

where

$$C_\epsilon=\max\left(1,\frac{1}{3}(1+\epsilon)^3,(1+\epsilon)\left(\frac{(3+\epsilon^2)^2}{2}+(1+\epsilon)\right),(1-\epsilon)(1+\epsilon)+2\right)$$

and

$$c_\epsilon=\min\left(\frac{1}{4},\frac{1}{6}-\epsilon P(\epsilon),1-2\epsilon-\epsilon^2\right),$$

where $P(\epsilon)$ is a polynomial in ϵ. ϵ is chosen such that the constant c_ϵ is positive. Using lemma 1.5, there exists u_0, negative, $|u_0|$ large enough, such that the estimates of the coordinates hold in $\Omega^+_{u_0}$ and, consequently, the equivalence is true on this domain.♦

u_0 is now fixed, being equal to the u_0 associated with the ϵ which ensures that the energy equivalence established in proposition 2.5 holds. The neighborhood of i^0 where the energy estimates are relevant is then $\Omega^+_{u_0}$.

2.1.2 Energy estimates near i^0

The energy estimates are established between $\Sigma_0^{u_0>}$, S_{u_0} and $\mathcal{I}^+_{u_0}$, by writing a Stokes theorem between $\mathcal{H}_s$, $S^s_{u_0}=\{(u,R,\omega_{\mathbb{S}^2})|u=u=u_0,u\leq -sr^*\}$ and $\mathcal{I}^+_{u_0}$.

The first step consists in evaluating the error term:

Lemma 2.6. *The error is given by:*

$$\hat{\nabla}^a\left(\hat{T}^bT_{ab}\right)=4mR^2(3+uR)\left(\partial_R\phi\right)^2+(1-12mR)\,\phi\left(u^2\partial_u\phi-2(1+uR)\partial_R\phi\right)+\hat{T}^a\hat{\nabla}_ab\frac{\phi^4}{4}.$$

Proof. The Killing form of the vector $\hat{T}^a$ is calculated via the Lie derivative of the metric:

$$
\begin{aligned}
\hat{\nabla}_{(a}\hat{T}_{b)} &= \mathrm{L}_{\hat{T}}\hat{g}\\
&= \mathrm{L}_{\hat{T}}(R^2(1-2mR))(\mathrm{d}u)^2 + 2R^2(1-2mR)\mathrm{L}_{\hat{T}}(\mathrm{d}u)\mathrm{d}u - 2\mathrm{L}_{\hat{T}}(\mathrm{d}u)\mathrm{d}R - 2\mathrm{L}_{\hat{T}}(\mathrm{d}R)\mathrm{d}u - 2\mathrm{L}_{\hat{T}}(\mathrm{d}\omega_{\mathbb{S}^2})\\
&= -4R(1+uR)(1-3mR)(\mathrm{d}u)^2 + 2R^2(1-2mR)\mathrm{d}\left(\mathrm{L}_{\hat{T}}(u)\right)\mathrm{d}u - 2\mathrm{d}\left(\mathrm{L}_{\hat{T}}(u)\right)\mathrm{d}R - 2\mathrm{d}\left(\mathrm{L}_{\hat{T}}(R)\right)\mathrm{d}u - 0\\
&= -4R(1+uR)(1-3mR)(\mathrm{d}u)^2 + 2R^2(1-2mR)\mathrm{d}(u^2)\mathrm{d}u - 2\mathrm{d}\left(u^2\right)\mathrm{d}R - 2\mathrm{d}\left(-2(1+uR)\right)\mathrm{d}u\\
&= -4R(1+uR)(1-3mR)(\mathrm{d}u)^2 + 4uR^2(1-2mR)(\mathrm{d}u)^2 - 4u\mathrm{d}u\mathrm{d}R + 4(R\mathrm{d}u + u\mathrm{d}R)\mathrm{d}u\\
&= (-2R(1+uR)(1-3mR) + 4uR^2(1-2mR) + 4R)(\mathrm{d}u)^2\\
&= (12mR^2 + 4muR^3)(\mathrm{d}u^2),
\end{aligned}
$$

or, in the vector form:

$$\hat{\nabla}^{(a}\hat{T}^{b)} = 4mR^2(3+uR)\partial_R\partial_R.$$

A direct consequence of the above formula is that the Killling form is trace-free:

$$
\begin{aligned}
\nabla^a\hat{T}_a &= 4mR^2(3+uR)\hat{g}(\partial_R,\partial_R)\\
&= 0.
\end{aligned}
$$

The scalar curvature of the rescaled metric is given by equation (1.2); choosing $\Omega = R$ gives:

$$
\begin{aligned}
\frac{1}{6}\mathrm{Scal}_{\hat{g}} &= R^3\nabla_b\nabla^b R\\
&= 2mR.
\end{aligned}
$$

Finally, the error term is given by:

$$
\begin{aligned}
\nabla^{(a}\hat{T}^{b)}T_{ab} &= 4mR^2(3+uR)\left(\partial_R\phi\right)^2 + (1-12mR)\left(\hat{T}^a\nabla_a\phi\right)\phi + \hat{T}^a\hat{\nabla}_a b\frac{\phi^4}{4}\\
&= 4mR^2(3+uR)\left(\partial_R\phi\right)^2 + (1-12mR)\,\phi\left(u^2\partial_u\phi - 2(1+uR)\partial_R\phi\right)\\
&\quad + \hat{T}^a\hat{\nabla}_a b\frac{\phi^4}{4}. \blacklozenge
\end{aligned}
$$

Remark 2.7. *As noticed in [63], one obstacle to the use of the parameter s for the foliation is the fact that this parametrization in s is not smooth in the sense that $(r^*)^{-1}$ is not a smooth function of R at $R=0$.*

In order to avoid this singularity, the speed of the identifying vector field is decreased by means of a change of variable: let τ be the function defined by:

$$\tau : \begin{array}{ccc} [0,1] & \longrightarrow & [0,2] \\ s & \longmapsto & -2(\sqrt{s}-1). \end{array} \tag{2.9}$$

$\mathscr{I}^+_{u_0}$ is then given by $s=0$ and $\tau=2$ and $\Sigma_0^{u_0>}$ is given by $s=1$ and $\tau=0$. The new identifying vector field V^a is then chosen such that:

$$\mathrm{d}\tau(V^a) = 1 \text{ so that } V^a = (r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{\frac{R}{|u|}}\partial_R^a. \tag{2.10}$$

The foliation $\mathcal{H}_s$, when parametrized by τ, is denoted by Σ_τ

Finally, we can prove the following estimates:

Proposition 2.8. *The following equivalence holds:*

$$E(\mathcal{I}^+_{u_0}) + E(S_{u_0}) \approx E(\Sigma_0^{u_0<})$$

where

$$E(\mathcal{I}^+_{u_0}) = \int_{\mathcal{I}^+_{u_0}} i^\star_{\mathcal{I}^+}(\star \hat{T}^a T_{ab}) \text{ and } E(S_{u_0}) = \int_{S_{u_0}} i^\star_{S_{u_0}}(\star \hat{T}^a T_{ab}).$$

Proof. The proof of these estimates relies on Stokes theorem applied between the hypersufaces $S^s_{u_0} = \{(u, R, \omega_{\mathbb{S}^2}) | u = u_0, |u| \geq sr^*\}$, $\mathcal{H}_{s(\tau)} = \Sigma_\tau$ and $\Sigma_0^{u_0>}$. Stokes theorem can be used here since the data are compactly supported in Σ_0 and, as a consequence, the future of the support of the initial data does not contain the singularity i^0. Let be $M^s_{u_0}$ be the subset of $\Omega^+_{u_0}$ whose boundary consists of these hypersufaces. We have:

$$\int_{S^{s(\tau)}_{u_0}} i^\star_{S_u}(\star \hat{T}^a T_{ab}) + \int_{\Sigma_\tau} i^\star_{\Sigma_\tau}(\star \hat{T}^a T_{ab}) - \int_{\Sigma_0^{u_0>}} i^\star_{\Sigma_0}(\star \hat{T}^a T_{ab}) - = \int_{M^{s(\tau)}_{u_0}} \hat{\nabla}^a \left(\hat{T}^b T_{ab} \right) \mu[\hat{g}]$$

and, using the notations in the proposition, the foliation given by τ defined by equation (2.9) and lemma 2.6, this becomes:

$$E(S^{s(\tau)}_{u_0}) + \int_{\Sigma_\tau} i^\star_{\Sigma_\tau}(\star \hat{T}^a T_{ab}) - \int_{\Sigma_0^{u_0>}} i^\star_{\Sigma_0}(\star \hat{T}^a T_{ab})$$

$$= \int_0^\tau \left(\int_{\Sigma_\tau} \left\{ 4mR^2(3+uR) \left(\partial_R\phi\right)^2 + (1-12mR)\,\phi \left(u^2\partial_u\phi - 2(1+uR)\partial_R\phi\right) \right.\right.$$

$$\left.\left. + \hat{T}^a \hat{\nabla}_a b \frac{\phi^4}{4} \right\} (r^* R)^{\frac{3}{2}} (1-2mR) \sqrt{\frac{R}{|u|}} \mathrm{d}u \wedge \mathrm{d}\omega_{\mathbb{S}^2} \right) \mathrm{d}\tau$$

The error term is bounded above in absolute value; each term is evaluated separetly:

$$\begin{aligned} &|(r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{\frac{R}{|u|}}4mR^2(3+uR)\left(\partial_R\phi\right)^2| \\ &= (r^*R)^{\frac{3}{2}}(1-2mR)(3+|u|R)(R|u|)^{1/2}R\frac{R}{|u|}\left(\partial_R\phi\right)^2 \\ &\leq (1+\epsilon)^{\frac{3}{2}} \cdot 1 \cdot (4+\epsilon) \cdot (1+\epsilon)\frac{\epsilon}{2m}\frac{R}{|u|}\left(\partial_R\phi\right)^2 \\ &\lesssim \frac{R}{|u|}\left(\partial_R\phi\right)^2. \end{aligned}$$

and

$$|\,(1-12mR)\,(r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{\frac{R}{|u|}}u^2\phi\partial_u\phi| = |\,(1-12mR)\,(r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{R|u|}\phi(u\partial_u\phi|)$$

$$\begin{aligned} &\lesssim \frac{1}{2}(1+6\epsilon)(1+\epsilon)^{\frac{3}{2}} \cdot 1 \cdot (1+\epsilon)\left(\phi^2 + (u\partial_u\phi)^2\right) \\ &\lesssim \phi^2 + (u\partial_u\phi)^2. \end{aligned}$$

The remaining term is controlled by:

$$\begin{aligned} |(1-12mR)(r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{\frac{R}{|u|}}\phi\partial_R\phi \quad &\leq \quad (1+6\epsilon)(1+\epsilon)^{\frac{3}{2}}\phi\left(\sqrt{\frac{R}{|u|}}\partial_R\phi\right) \\ &\leq \quad (1+6\epsilon)(1+\epsilon)^{\frac{3}{2}}\left(\phi^2 + \frac{R}{|u|}(\partial_R\phi)^2\right). \end{aligned}$$

Remark 2.9. *This term is the main obstacle in the use of the parameter s: if the foliation was parametrized by s, this term would be replaced by:*

$$\left|(1-12mR)\frac{(r^*R)^2(1-2mR)}{|u|}\phi\partial_R\phi\right| \leq (1+6\epsilon)(1+\epsilon)^2\left|\frac{\phi\partial_R\phi}{u}\right|$$

which cannot be compared to $\phi^2 + \frac{R}{|u|}(\partial_R\phi)^2$.

Gathering these inequalities, it remains:

$$\left|\int_0^\tau \left(\int_{\Sigma_\tau}\left\{4mR^2(3+uR)\left(\partial_R\phi\right)^2 + (1-12mR)\,\phi\left(u^2\partial_u\phi - 2(1+uR)\partial_R\phi\right)\right.\right.\right.$$
$$\left.\left.\left.+\hat{T}^a\hat{\nabla}_a b\frac{\phi^4}{4}\right\}(r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{\frac{R}{|u|}}\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\right)\mathrm{d}\tau\right| \lesssim \int_0^\tau E(\Sigma_r)\mathrm{d}r.$$

Finally, the following inequalities hold:

$$\begin{array}{lcl} E(\Sigma_\tau) + E(S_{u_0}^{s(\tau)}) & \lesssim & \int_0^\tau E(\Sigma_r)\mathrm{d}r + E(\Sigma_0^{u_0>}) \\ E(\Sigma_0^{u_0>}) & \lesssim & \int_0^\tau E(\Sigma_r)\mathrm{d}r + E(\Sigma_\tau) + E(S_{u_0}^{s(\tau)}) \end{array} \tag{2.11}$$

Since $E(S_{u_0}^{s(\tau)})$ is positive, the integral inequality holds:

$$E(\Sigma_\tau) \lesssim \int_0^\tau E(\Sigma_r)\mathrm{d}r + E(\Sigma_0^{u_0>}).$$

Using Gronwall's lemma, we obtain:

$$E(\Sigma_\tau) \lesssim E(\Sigma_0^{u_0>}). \tag{2.12}$$

Putting this inequality back into (2.11), we obtain the first part of the inequality, for $\tau = 2$:

$$E(\mathcal{I}_{u_0}^+) + E(S_{u_0}) \lesssim E(\Sigma_0^{u_0>}).$$

The other inequality is obtained by doing the same calculation between the hypersurfaces $S_{u_0,s} = \{(u,R,\omega_{\mathbb{S}^2}) | u = u_0, |u| \leq sr^*\}$, $\mathcal{H}_s$ and $\Sigma_0^{u_0>}$. Let be $M_s^{u_0}$ be the subset of $\Omega_{u_0}^+$ whose boundary consists of these hypersurfaces. Applying Stokes theorem, using the parametrization by τ and the previous estimates of the error term, we obtain:

$$E(\Sigma_\tau) \lesssim \int_0^\tau E(\Sigma_r)\mathrm{d}r + E(\mathcal{I}_{u_0}^+) + E(S^{u_0,s(\tau)})$$

As a consequence, since the integrand in $E(S^{u_0,s(\tau)})$ is positive, the following integral inequality holds:

$$E(\Sigma_\tau) \lesssim \int_0^\tau E(\Sigma_r)\mathrm{d}r + E(\mathcal{I}_{u_0}^+) + E(S^{u_0}).$$

The use of Gronwall lemma gives the second inequality:

$$E(\Sigma_0^{u_0<}) \lesssim E(\mathcal{I}_{u_0}^+) + E(S^{u_0})\blacklozenge$$

2.2 Energy estimates far from the spacelike infinity i^0

The estimates which are obtained in this section are widely inspired by the work of Hörmander [53] and generalized in [73] to establish the existence of solutions for the characteristic Cauchy problem for the wave equation on a curved background. The main tool consists in writing the characteristic, or weakly characteristic (that is to say is locally either spacelike or degenerate; this is also referred to as achronal) surface as the graph of a function and expressing all the relevant quantities in term of this graph. This method was also used by Mason-Nicolas in [62] to control how spacelike surfaces converge to null infinity.

$\hat{M}\backslash\Omega^+_{u_0} \cap J^+(\Sigma_0)$ is divided in two parts as follows:

- let Σ be a spacelike hypersurface in $\hat{M}$ for the metric $\hat{g}$ such that $\Sigma \cap \mathscr{I}^+ = S_{u_0} \cap \mathscr{I}^+$ and Σ is orthogonal to $\hat{T}^a$.
- the part of $\hat{M}\backslash\Omega^+_{u_0}$ contained between Σ_0 and Σ, denoted by V;
- and finally the future of Σ, containing i^+, U. The subset of its boundary in $\mathscr{I}^+$ is denoted by $\mathscr{I}^+_T$.

This decomposition of the future of Σ_0 is represented in figure 3.2.

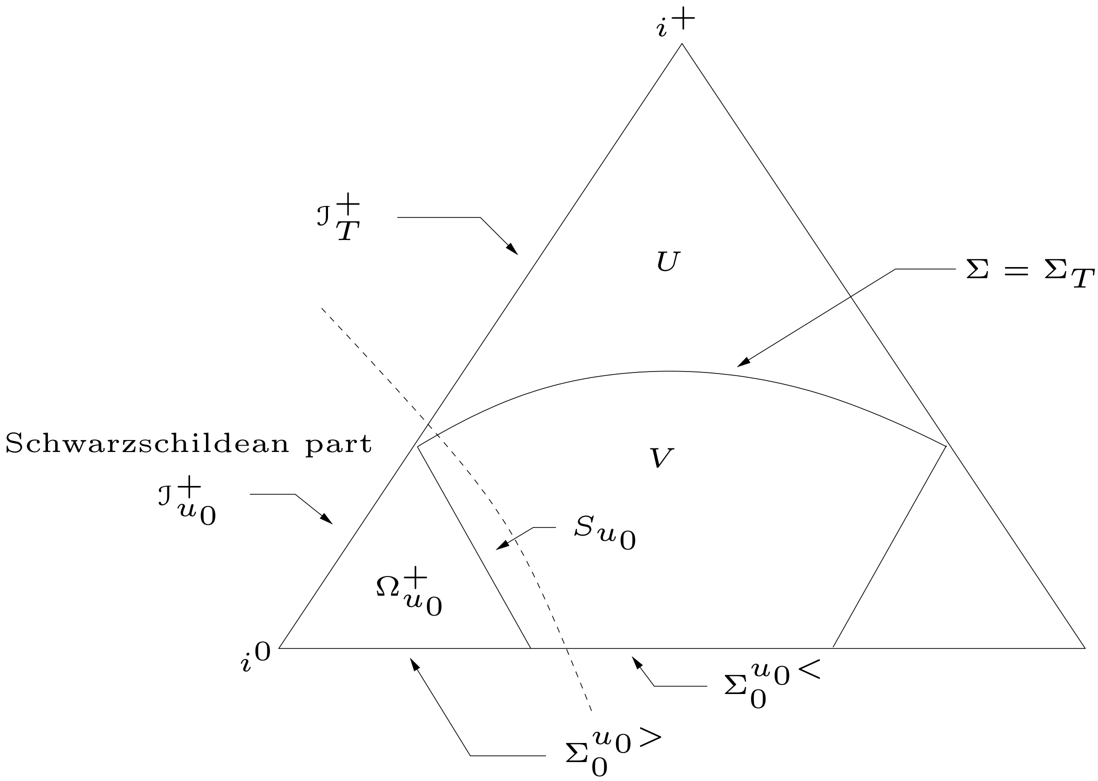


Figure 2.2: Future of Σ_0

Remark 2.10. *The fact that the timelike vector field $\hat{T}^a$ is orthogonal to the spacelike hypersurface is an assumption which is made about this timelike vector field. It is build outside the Schwarzshildean part of the manifold as follows:*

- *outside $\Omega^+_{u_0}$, $\hat{T}^a$ is smoothly extended such that it is orthogonal to a uniformly spacelike hypersurface Σ whose boundary is $\mathcal{I}^+ \cap S_{u_0}$.*
- *The intersection of S_{u_0} and $\mathcal{I}^+$ is in the Schwarzschildean part and is given by $\{u = u_0, R = 0\}$. The vector field $\hat{T}^a$ is then orthogonal to this 2-dimensional surface.*

This subsection deals with estimates on U. As previously said, we use here Hörmander's techniques which consists in writing $\mathcal{I}$ as the graph of a function.

We consider on U the flow associated with $e_0^a = \frac{\hat{T}^a}{\hat{g}_{cd}\hat{T}^c\hat{T}^d}$. Let t be the time function induced by this flow. The set $\{e_0^a\}$ is completed in an orthonormal basis of $\hat{M}$ by choosing an orthonormal basis $\{e_i^a; i = 1, 2, 3\}$ of the spacelike foliation $\{\Sigma_t\}$ induced by t. For the sake of clarity, Σ is denoted Σ_T as corresponding to the slice $\{t = T\}$ (T is chosen to be non zero, in order not to introduce confusion with Σ_0).

2.2.1 Geometric description of $\mathcal{I}^+_T$

Using the flow Ψ_t associated with e_0^a, $\mathcal{I}^+_T$ can be identified with Σ_T:

$$\begin{array}{ccc} \Sigma_T & \longrightarrow & \mathcal{I}^+_T \\ x & \longmapsto & \Psi_{\varphi(x)}(x) \end{array} \tag{2.13}$$

where $\varphi(x)$ is the time at which the curve $t \mapsto \Psi_t(x)$ hits $\mathcal{I}^+_T$. $\mathcal{I}^+_T$ can then be considered as defined by the graph of the function $\varphi : x \in \Sigma_T \longmapsto \varphi(x)$.

We denote by $\nabla_i\varphi$ the derivatives of φ with regard to the vector tangent to Σ_T $e_{\mathbf{i}}^a$ at time T.

Remark 2.11. *1. As noticed in the introduction, the spacetimes constructed by Chrusciel-Delay and Corvino-Schoen have the specificity that the regularity at $\mathcal{I}^+\backslash\{i^0, i^+\}$ can be specified arbitrarily. In order to insure that some geometric quantities are defined, we assumed that the manifold is C^2 differentiable at $\mathcal{I}^+\backslash\{i^0, i^+\}$. The use of the implicit functions theorem then insures that the function φ has the same regularity.*

2. The function φ is defined on a compact set and as such admits a maximum. This maximum is denoted by T_{max}.

The lapse function associated with this choice of time t is denoted by N. The metric can be decomposed as:

$$\hat{g} = N^2(\mathrm{d}t)^2 - h_{\Sigma_t}$$

where h is a Riemannian metric on Σ_t depending on the spacelike leaves of the foliation induced by the time function, and N is the (positive) lapse function. Since the time function is build from the vector e_0^a, the vector field ∂_t satisfies:

$$\partial_t = N e_0^a.$$

The following lemma describes the geometry of $\mathcal{I}^+_T$ in term of the parametrization:

Lemma 2.12. *The vector N^a defined by*

$$N^a = e_0^a - \sum_{j\in\{1,2,3\}} N\nabla_j\varphi e_{\boldsymbol{j}}^a$$

is normal and tangent to the hypersurface $\mathcal{I}_T^+$.
The set of vectors defined by, for $i \in \{1,2,3\}$,

$$t_{\boldsymbol{i}}^a = N\nabla_i\varphi e_0^a - e_{\boldsymbol{i}}^a$$

are normal to N^a and, as such, forms a basis of $T\mathcal{I}_T^+$.

Proof. The fact that N^a is null directly comes from the fact N^a is normal to $\mathcal{I}_T^+$, which is a null surface. The derivatives of φ then satisfies:

$$N^2 \sum_{i=1,2,3} (\nabla_i\varphi)^2 = 1$$

The tangent plane to $\mathcal{I}_T^+$ is given by the kernel of the differential of the application

$$x \longmapsto (\varphi(x), x)$$

which is given by

$$h^a \in \Sigma_T \longmapsto h^a + g_{ij}(\varphi(x), x)\nabla^i\varphi h^j \underbrace{Ne_0^a}_{\partial_t}.$$

It is then clear that the set of vectors

$$t_{\mathbf{i}}^a = N\nabla^i\varphi e_0^a - e_{\mathbf{i}}^a$$

forms a basis of $T\mathcal{I}_T^+$. N^a is then a linear combination of them:

$$N^a = \sum_{i=1,2,3} N\nabla^i\varphi t_{\mathbf{i}}^a. \blacklozenge$$

As a direct consequence of this lemma, the vector defined by

$$\tau^a = e_0^a + \sum_{j\in\{1,2,3\}} N\nabla_j\varphi e_{\mathbf{j}}^a$$

is null and transverse to $\mathcal{I}_T^+$.

In order to complete the geometric description of $\mathcal{I}_T^+$ and facilitate the calculation afterwards, we introduce the following objects:

- using the Geroch-Held-Penrose formalism, the set of two null vectors (τ^a, N^a) is completed by two normalized spacelike vectors (v_1^a, v_2^a) tangent to $\mathcal{I}_T^+$ to form a basis of $T\hat{M}$;
- $\mathcal{I}_T^+$ is endowed with the 3-form:

$$\mu_{\mathcal{I}} = t_a^1 \wedge t_a^2 \wedge t_a^3$$

which satisfies:

$$\begin{aligned}\tau_a \wedge t_a^1 \wedge t_a^2 \wedge t_a^3 &= (1 + N^2 \sum_{i=1,2,3} (\nabla^i\varphi)^2)\mu[\hat{g}] \\ &= 2\mu[\hat{g}];\end{aligned}$$

this 3-form will be used as the form of reference to calculate the energy on $\mathcal{I}_T^+$.

Remark 2.13. *The tangent vector n^a used in definition 1.12 for the H^1-norm on $\mathscr{I}^+$ is colinear to the vector N^a:*

$$N^a = \frac{n^a}{\hat{g}_{cd}\hat{T}^c\hat{T}^d}.$$

As a consequence, the norm used in definition 1.12 are equivalent.

Finally, in order to prepare the estimates, the expression of the 3-form $\star\hat{T}^a T_{ab}$ on the surfaces Σ_t and $\mathscr{I}^+_T$ is given:

Lemma 2.14. *The restrictions of the energy 3-form $\star\hat{T}^a T_{ab}$ to Σ_t, for t given in $[T, T_{max}]$, and $\mathscr{I}^+_T$ are given by, respectively:*

$$i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) = ||\hat{T}^a|| \left(\frac{1}{2}\sum_{i=0}^{4}(e^a_{\mathbf{i}}\nabla_a\phi)^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right) e^1_a \wedge e^2_a \wedge e^3_a$$

and

$$i^\star_{\mathscr{I}^+_T}\left(\star\hat{T}^a T_{ab}\right) = \frac{||\hat{T}^a||}{4}\left((N^a\hat{\nabla}_a\phi)^2 + (v^a_1\hat{\nabla}_a\phi)^2 + (v^a_2\hat{\nabla}_a\phi)^2) + \frac{\phi^2}{2}\right) t^1_a \wedge t^2_a \wedge t^3_a$$

Remark 2.15.

1. *The expression which is given for the energy form on $\mathscr{I}$ is consistent with the one obtain by Hörmander, since it only depends on tangential derivatives to null infinity. It is nonetheless not identical: the result of Hörmander is similar to a calculation made with respect to the Riemannian metric obtained from $\hat{g}$ and the timelike vector field $\hat{T}^a$.*
2. *The part of the energy form on $\mathscr{I}$ given by $(v^a_1\hat{\nabla}_a\phi)^2 + (v^a_2\hat{\nabla}_a\phi)^2$ is usually interpreted as the norm of the gradient on a 2 sphere, even though the distribution of 2-planes $Span(v_1, v_2)$ is not integrable.*

Proof. Using the basis $(e^a_{\mathbf{i}})_{i=0,\ldots,4}$ which is adapted to the foliation, the energy 3-form over Σ_t can easily be calculated:

$$i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) = \left(\hat{T}^a\hat{\nabla}_a\phi\right) i^\star_{\star\Sigma_t}(\hat{\nabla}_b\phi) + \left(-\frac{1}{2}\hat{g}_{cd}\hat{\nabla}^c\phi\hat{\nabla}^d\phi + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right) i^\star_{\Sigma_t}(\star T_b)$$

Since the vectors $\{e^a_{\mathbf{i}}\}_{i=1,2,3}$ are tangent to the hypersufaces Σ_t, we obtain:

- $i^\star_{\Sigma_t}(\star T_a) = ||\hat{T}^a|| e^a_0 \lrcorner \mu[\hat{g}] = ||\hat{T}^a|| e^1_a \wedge e^2_a \wedge e^3_a$
- $i^\star_{\Sigma_t}(\hat{\nabla}_b\phi) = e^b_0\nabla_b\phi\,(e^a_0 \lrcorner \mu[\hat{g}]) = (e^b_0\nabla_b\phi) e^1_a \wedge e^2_a \wedge e^3_a$

and

$$i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) = ||\hat{T}^a|| \left(\frac{1}{2}\sum_{i=0}^{4}(e^a_{\mathbf{i}}\nabla_a\phi)^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right) e^1_a \wedge e^2_a \wedge e^3_a.$$

To calculate the restriction of the energy form to $\mathscr{I}^+_T$, the vector fields $\hat{T}^a$ and $\hat{\nabla}\phi$ are split over the basis $(N^a, \tau^a, v^a_1, v^a_2)$:

$$\hat{T}^b = \frac{||\hat{T}^b||}{2}\left(N^b + \tau^b\right) \tag{2.14}$$

$$\hat{\nabla}^b\phi = \frac{N^a\hat{\nabla}_a\phi}{2}\tau^b + \frac{\tau^a\hat{\nabla}_a\phi}{2}N^b - (v^a_1\hat{\nabla}_a\phi)v^b_1 + v^a_2\hat{\nabla}_a\phi v^b_2$$

$$\hat{\nabla}_c\phi\hat{\nabla}^c\phi = N^a\hat{\nabla}_a\phi\tau^a\hat{\nabla}_a\phi - (v^a_1\hat{\nabla}_a\phi)^2 - (v^a_2\hat{\nabla}_a\phi)^2. \tag{2.15}$$

The only relevant terms in the expressions of $i^\star_{\mathcal{I}^+}(\star\hat{\nabla}_b\phi)$ and $i^\star_{\mathcal{I}^+}(\hat{T}_b)$ are those which are transverse to $\mathcal{I}$, so that:

$$\begin{array}{rclcrcl} i^\star_{\mathcal{I}^+}(\hat{T}_b) & = & \frac{||\hat{T}^a||}{2} i^\star_{\mathcal{I}}(\tau_a) & \quad & i^\star_{\mathcal{I}^+}(\star\hat{\nabla}_b\phi) & = & \frac{N^a\hat{\nabla}_a\phi}{2} i^\star_{\mathcal{I}}(\tau_a) \\ & = & \frac{||\hat{T}^a||}{2} \tau^a \lrcorner \mu[\hat{g}] & & & = & \frac{N^a\hat{\nabla}_a\phi}{2} \tau^a \lrcorner \mu[\hat{g}] \\ & = & \frac{||\hat{T}^a||}{4} t^1_a \wedge t^2_a \wedge t^3_a & & & = & \frac{N^a\hat{\nabla}_a\phi}{4} t^1_a \wedge t^2_a \wedge t^3_a \end{array}$$

and, finally, using equations (2.14) and (2.15), since the function b vanishes at $\mathcal{I}$:

$$\begin{array}{rcl} i^\star_{\mathcal{I}^+_T}\left(\star\hat{T}^a T_{ab}\right) & = & \left(e^a_0\hat{\nabla}_a\phi N^a\hat{\nabla}_a\phi - \frac{1}{2}\hat{\nabla}_c\phi\hat{\nabla}^c\phi + \frac{\phi^2}{2}\right)\frac{||\hat{T}^a||}{4} t^1_a \wedge t^2_a \wedge t^3_a \\ & = & \frac{||\hat{T}^a||}{8}\left((N^a\hat{\nabla}_a\phi)^2 + (v^a_1\hat{\nabla}_a\phi)^2 + (v^a_2\hat{\nabla}_a\phi)^2) + \phi^2\right) t^1_a \wedge t^2_a \wedge t^3_a \blacklozenge \end{array}$$

2.2.2 Energy estimates on U

The techniques used in this section are exactly the same as in the other section: they rely on Gronwall lemma and Stokes theorem carefully applied to the 3-form $\star\hat{T}^a T_{ab}$.

The first step of the proof consists in establishing a decay result for the energy on slices $\{t = constant\}$.

Proposition 2.16. *Let $E(\Sigma_t)$ be the energy on the slice Σ_t:*

$$E(\Sigma_t) = \int_{\Sigma_t}\left(\frac{1}{2}\sum_{i=0}^{4}(e^a_i\nabla_a\phi)^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\mu_{\Sigma_t}$$

where $\mu_{\Sigma_t} = e^1_a \wedge e^2_a \wedge e^3_a$.
Then this energy satisfies:

$$E(\Sigma_t) \approx \int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right)$$

and for s and t two real numbers in $[T, T_{max}]$, such that $t \geq s$:

$$E(\Sigma_t) \lesssim E(\Sigma_s)$$

Remark 2.17. *1. In this section, the calculations are made with respect to $E(\Sigma_T)$ rather than $\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right)$.*

Proof. Since $\hat{T}^a$ is a non-vanishing timelike vector field on the compact $\hat{M}$, there exists a positive constant C such that:

$$\frac{1}{C} \leq ||\hat{T}^a|| \leq C$$

and, as a consequence of lemma 2.14, the energy $E(\Sigma_t)$ is equivalent to $\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right)$ since:

$$\frac{1}{2C}E(\Sigma_t) \leq \int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) \leq CE(\Sigma_t).$$

Remark 2.18. *The same result holds for the energy on $\mathscr{I}^+_T$, that it to say that:*

$$\int_{\mathscr{I}^+_T} i^\star_{\mathscr{I}^+_T}\left(\star\hat{T}^a T_{ab}\right) \approx \int_{\mathscr{I}^+_T}\left((N^a\hat{\nabla}_a\phi)^2 + (v_1^a\hat{\nabla}_a\phi)^2 + (v_2^a\hat{\nabla}_a\phi)^2) + \frac{\phi^2}{2}\right) t^1_a \wedge t^2_a \wedge t^3_a$$

We denote by $E(\mathscr{I}^+_T)$ the right-hand side of this equation. The expression of this energy is not intrisic, since it depends on the one hand, on the parametrization of $\mathscr{I}^+$ by the function φ and, on the other hand, on the choice of a basis. The energy used by Hörmander suffers from the same property that it depends on the graph and on the chosen coordinate system.

We assume here that $t > s$ and apply Stokes theorem between the surfaces Σ_t and Σ_s. The part of $\mathscr{I}^+$ between the time t and s is denoted $\mathscr{I}^t_s$ and the part of U between Σ_t and Σ_s, U^t_s. The following equality holds:

$$\begin{aligned}&\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) + \int_{\mathscr{I}^t_s} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) - \int_{\Sigma_s} i^\star_{\Sigma_s}\left(\star\hat{T}^a T_{ab}\right)\\ &= \int_{U^t_s}\left(\hat{\nabla}^{(a}\hat{T}^{b)}T_{ab} + (1-\frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi\hat{T}^a\hat{\nabla}_a\phi + \hat{T}^a\hat{\nabla}_a\frac{\phi^4}{4}\right)\mu[\hat{g}]\end{aligned}$$

As it was noticed in lemma 2.14, the integral over $\mathscr{I}^+$ of the energy 3-form restricted to $\mathscr{I}$ is positive. So the following inequalities holds:

$$\begin{aligned}&\left|\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) + \int_{\mathscr{I}^t_s} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) - \int_{\Sigma_s} i^\star_{\Sigma_s}\left(\star\hat{T}^a T_{ab}\right)\right|\\ &\leq \int_{U^t_s}\left|\left(\hat{\nabla}^{(a}\hat{T}^{b)}T_{ab} + (1-\frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi\hat{T}^a\hat{\nabla}_a\phi + \hat{T}^a\hat{\nabla}_a b\frac{\phi^4}{4}\right)\right|\mu[\hat{g}]\end{aligned}$$

and, as a consequence,

$$\begin{aligned}&\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) + \int_{\mathscr{I}^t_s} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right)\\ &\leq \int_{U^t_s}\left|\hat{\nabla}^{(a}\hat{T}^{b)}T_{ab}\right| + \left|(1-\frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi\hat{T}^a\hat{\nabla}_a\phi\right| + \left|\hat{T}^a\hat{\nabla}_a b\right|\frac{\phi^4}{4}\mu[\hat{g}] + \int_{\Sigma_s} i^\star_{\Sigma_s}\left(\star\hat{T}^a T_{ab}\right).\end{aligned}$$

Since $\int_{\mathscr{I}^t_s} i^\star_{\Sigma_t}$ is non-negative (see remark 2.18) and

$$\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) \approx E(\Sigma_r),$$

it remains:

$$E(\Sigma_t) \lesssim \int_{U^t_s}\left|\hat{\nabla}^{(a}\hat{T}^{b)}T_{ab}\right| + \left|(1-\frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi\hat{T}^a\hat{\nabla}_a\phi\right| + \left|\hat{T}^a\hat{\nabla}_a b\right|\frac{\phi^4}{4}\mu[\hat{g}] + E(\Sigma_s).$$

Since $\overline{U}$ is compact, there exists a contant c depending on $\hat{\nabla}^a\hat{T}^b$, $\mathrm{Scal}_{\hat{g}}$ and $\hat{T}^a\hat{\nabla}_a b$ which controls each term in the error

$$\int_{U^t_s}\left(\hat{\nabla}^{(a}\hat{T}^{b)}T_{ab} + (1-\frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi\hat{T}^a\hat{\nabla}_a\phi + \hat{T}^a\hat{\nabla}_a b\frac{\phi^4}{4}\right)\mu[\hat{g}]$$

in function of the energy on a slice at time r. To perform such an estimate, the 2-form $\hat{\nabla}^{(a}\hat{T}^{b)}$ is split over the orthonormal basis $(e_{\mathbf{i}}^a)_{i=0,\dots,4}$; each of the components is bounded by c. The remaining terms, when contracting with T_{ab}, are of either products of derivatives or functions which can be estimated by ϕ^2 or $b\phi^4$ and their derivatives.

Finally, the volume form $\mu[\hat{g}]$ is decomposed over the basis $(e_{\mathbf{i}}^a)_{i=0,\dots,4}$, to obtain:

$$E(\Sigma_t) \lesssim \int_s^t E(\Sigma_r)\mathrm{d}r + E(\Sigma_s)$$

where the form $\mathrm{d}r$ is e_a^0. So, applying Gronwall lemma, we obtain since we are working in finite time:

$$E(\Sigma_t) \lesssim E(\Sigma_s).\blacklozenge$$

A straightforward consequence of this proposition is that all the energies on slices are controlled by the energy on Σ_T. This is a necessary step to establish the following proposition:

Proposition 2.19. *The energies on $\mathcal{I}_T^+$ and on Σ_T are equivalent:*

$$E(\mathcal{I}_T^+) \approx E(\Sigma_T).$$

Proof. The proof is based on the use of Stokes theorem. We denote by, for t between T and T_{max}:

- U_t the part of U for time greater than t;
- $\mathcal{I}_t^+$ the part of $\mathcal{I}^+$ for time greater than t.

Stokes theorem is used between the hypersurfaces $\mathcal{I}_t^+$ and Σ_t:

$$\begin{aligned}&\int_{\mathcal{I}_t^+} i^\star_{\mathcal{I}^+}\left(\star\hat{T}^a T_{ab}\right) - \int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right)\\ &= \int_{U_t}\left(\hat{\nabla}^{(a}\hat{T}^{b)}T_{ab} + (1-\frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi\hat{T}^a\hat{\nabla}_a\phi + \hat{T}^a\hat{\nabla}_a\frac{\phi^4}{4}\right)\mu[\hat{g}]\end{aligned}$$

Using exactly the same estimate as in proposition 2.16 of the error term, we obtain:

$$\left|\int_{\mathcal{I}_t^+} i^\star_{\mathcal{I}^+}\left(\star\hat{T}^a T_{ab}\right) - \int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right)\right| \lesssim \int_t^{T_{max}} E(\Sigma_r)\mathrm{d}r.$$

As a consequence, the two following inequalities hold:

$$\int_{\mathcal{I}_t^+} i^\star_{\mathcal{I}^+}\left(\star\hat{T}^a T_{ab}\right) - \int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) \lesssim \int_t^{T_{max}} E(\Sigma_r)\mathrm{d}r. \tag{2.16}$$

and

$$\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) - \int_{\mathcal{I}_T^+} i^\star_{\mathcal{I}^+}\left(\star\hat{T}^a T_{ab}\right) \lesssim \int_t^{T_{max}} E(\Sigma_r)\mathrm{d}r, \tag{2.17}$$

since

$$\int_{\mathcal{I}_T^+} i^\star_{\mathcal{I}^+}\left(\star\hat{T}^a T_{ab}\right) \geq \int_{\mathcal{I}_t^+} i^\star_{\mathcal{I}^+}\left(\star\hat{T}^a T_{ab}\right).$$

We first deal with inequality (2.16). Since, according to proposition 2.16, all the energies on a slice $\{t = \textit{constant}\}$ are controlled by $E(\Sigma_T)$ for $t \geq T$, the integral $\int_t^{T_{max}} E(\Sigma_s)\mathrm{d}r$ satisfies:

$$\int_t^{T_{max}} E(\Sigma_r)\mathrm{d}r \lesssim E(\Sigma_t).$$

Using inequality (2.16), a straightforward consequence is:

$$\int_{\mathcal{I}_T^+} i^\star_{\mathcal{I}^+}\left(\star \hat{T}^a T_{ab}\right) \lesssim E(\Sigma_T)$$

and with remark 2.18:

$$E(\mathcal{I}_T^+) \lesssim E(\Sigma_T).$$

On the other hand, to obtain the second inequality, we use Gronwall lemma in inequality (2.17); this gives:

$$E(\Sigma_t) \lesssim \int_{\mathcal{I}^+} i^\star_{\mathcal{I}^+}\left(\star \hat{T}^a T_{ab}\right)$$

and, consequently, for $t = T$:

$$E(\Sigma_T) \lesssim E(\mathcal{I}_T^+).\blacklozenge$$

2.3 Estimates on V

The geometric situation in this section is almost the same as in the previous one since the hypersurface S_{u_0} is known to be null, the only difference being that an additional term comes from the boundary of the future of $\Sigma_0^{u_0<}$. The energy estimates will then be obtained in exactly the same way. Nonetheless, we wish to keep the term with an energy on the hypersurface S_{u_0}, in order to compare these terms with the inequalities obtained in section 2.1.

It is clear that the time function which was defined in the previous section cannot be used here since $\hat{T}^a$ is not necessarily orthogonal to Σ_0. We now consider another time function τ, defined in $\overline{V}$ (or in a neighborhood of V, as done in remark 2.20 below) such that the hypersurface $\{\tau = 0\}$ corresponds to Σ_0 and the hypersurface $\{\tau = 1\}$ to $\Sigma = \Sigma_T$. We consider on V the orthonormal basis $(e_0^a, e_{\mathbf{i}}^a)_{i=1,2,3}$ such that:

$$e_0^a = \frac{\nabla^a \tau}{\hat{g}_{cd}\nabla^c\tau\nabla^d\tau}.$$

By construction, the vector fields $(e_{\mathbf{i}}^a)_{i=1,2,3}$ are tangent to the time slices.

We introduce the following function:

$$\alpha = 1 + \sum_{i=1,2,3} (\hat{g}_{cd} f^c e_{\mathbf{i}}^d)^2 \geq 1$$

where f^c is the normalization with respect to the metric $\hat{g}$ of the vector field $\hat{T}^a$.

Remark 2.20. • *The constant α is used to construct the so-called Lipschitz norm of the foliation Σ_t.*

- *Such a time function τ can be constructed as follows: since $\hat{M}$ is globally hyperbolic, there exists a time function on $\hat{M}$. Let $\tilde{\tau}$ be a time function. Let $\Psi_{\tilde{\tau}}$ be the flow associated with τ and let V_0 be the preimage of Σ on Σ_0 by the flow. We then obtain a diffeomorphism defined by:*

$$\begin{array}{ccc} V_0 & \longrightarrow & \Sigma \\ x & \longmapsto & \Psi_{\phi(x)}(x) \end{array}$$

  where $\phi(x)$ is the time at which the curve $\tilde{\tau} \mapsto \Psi_{\tilde{\tau}}(x)$ hits Σ. The new time function τ is then defined as: let p be a point lying between V_0 and Σ, p is written $\Psi_{\tilde{\tau}}(x)$; then

$$\tau(p) = \frac{\tilde{\tau}(p)}{\phi(x)}$$

  satisfies the required assumption.

- *$\hat{T}^a$ and e_0^a are both uniformly timelike. Therefore, the scalar product*

$$\beta = \hat{g}_{cd}\hat{f}^c e_0^a$$

  defines a positive function over $\overline{V}$ since T^a and e_0^a are both future directed and timelike over a compact set.

The following notations will be used, in order to be coherent with section 2.1:

- the section of the initial data surface below u_0 is denoted $\Sigma_0^{u_0<}$;
- as previously introduced, the slices $\{t = \textit{contant}\}$ in V are denoted Σ_t;
- the part V between time t and s (with $t < s$) is denoted by V_t^s;
- the part of S_{u_0} between time t and s (with $t < s$) is denoted by S_t^s;

The expression of the energy on S_{u_0} are the same as the one define in 2.1 for (see proposition 2.3). Since we are not working with an orthonormal basis, the expression of the energy is adapted to this surface.

Following the method already used in this paper, a geometric description of of the energy 3-form is given and an equivalence result of the integral of the 3-form with an well chosen energy is established:

Proposition 2.21. *The restriction of the energy 3-form to Σ_t is given by:*

$$i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) = \left\{\frac{(f^c\hat{\nabla}_c\phi)^2}{2(1+\sum_{i=1,2,3}(\hat{g}_{cd}f^c e^d c_i)^2)}\right.$$
$$\left. + \frac{1}{2}\left(\sum_{i=1,2,3}\left(1 - \frac{(\hat{g}_{cd}f^c e^d c_i)^2}{1+\sum_{i=1,2,3}(\hat{g}_{cd}f^c e^d c_i)^2}\right)(e_i^a\hat{\nabla}_a\phi)^2\right) + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right\}\frac{\hat{g}_{cd}\hat{T}^c\hat{T}^d}{\hat{g}_{cd}\hat{T}^c e_0^d} e_1^a \wedge e_2^a \wedge e_3^a$$

and, as a consequence, the following equivalence holds, for all t in $[0,1]$:

$$\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^a T_{ab}\right) \approx \int_{\Sigma_t}\left((f^c\hat{\nabla}_c\phi)^2 + \sum_{i=1,2,3}(e_i^a\hat{\nabla}_a\phi)^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right) e_1^a \wedge e_2^a \wedge e_3^a.$$

We denote by $E(\Sigma_t)$ this energy:

$$E(\Sigma_t) = \int_{\Sigma_t} \left((f^c \hat{\nabla}_c \phi)^2 + \sum_{i=1,2,3} (e_{\mathbf{i}}^a \hat{\nabla}_a \phi)^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4} \right) e_1^a \wedge e_2^a \wedge e_3^a.$$

Remark 2.22. *This proposition together with proposition 2.16 states that the energy on a spacelike slice for two uniformly timelike (for the metric $\hat{g}$) vector fields are equivalent. This justifies that we write in the same way the energy in proposition 2.16 and in this proposition.*

Proof. the strategy of the proof is the same as usual: the geometric objects are split over the basis $(f^a, e_{\mathbf{i}}^a)_{i=1,2,3}$ where f^a is transverse to Σ_t and $(e_i^a)_{i=1,2,3}$ tangent to Σ_t.

Considering the 4-form

$$f_a \wedge e_1^a \wedge e_2^a \wedge e_3^a$$

f^a is decomposed as follows:

$$f^a = \beta e_0^a - \sum_{i=1,2,3} \delta^{\mathbf{i}} e_{\mathbf{i}}^a$$

with

$$\delta^{\mathbf{i}} = \hat{g}_{cd} f^c e_{\mathbf{i}}^d \text{ and } \beta^2 = 1 + \sum_{i=1,2,3} (\delta^{\mathbf{i}})^2.$$

The volume form $\mu[\hat{g}]$ satisfies:

$$\mu[\hat{g}] = \frac{1}{\beta} f_a \wedge e_1^a \wedge e_2^a \wedge e_3^a$$

and its contraction with the vector f^a is:

$$f^a \lrcorner \mu[\hat{g}] = \left(\beta e_0^a - \sum_{i=1,2,3} \delta^{\mathbf{i}} e_{\mathbf{i}}^a \right) \lrcorner \mu[\hat{g}] = \beta e_1^a \wedge e_2^a \wedge e_3^a$$

as a consequence, we have:

$$i^\star_{\Sigma_t}(\star \hat{T}_a) = ||\hat{T}^a|| i^\star_{\Sigma_t}(\star f_a) = ||\hat{T}^a|| f^a \lrcorner \mu[\hat{g}] = ||\hat{T}^a|| \beta e_1^a \wedge e_2^a \wedge e_3^a.$$

We then deal with $\hat{\nabla}^c \phi$ which can be written:

$$\hat{\nabla}^c \phi = b f^c - \sum_{i=1,2,3} a^{\mathbf{i}} e_{\mathbf{i}}^c$$

where

$$b = \frac{\hat{g}_{ab} \hat{\nabla}^a \phi e_0^b}{\beta} = \frac{f^a \hat{\nabla}_a \phi + \sum_{i=1,2,3} \delta^{\mathbf{i}} e_{\mathbf{i}}^a \hat{\nabla}_a \phi}{\beta^2} \text{ and } a_{\mathbf{i}} = e_{\mathbf{i}}^a \hat{\nabla}_a \phi - b\delta^{\mathbf{i}} \text{ for } i \in \{1,2,3\}.$$

Consequently, its norm is:

$$\hat{\nabla}^c \phi \hat{\nabla}_c \phi \quad = \quad \frac{(f^a \hat{\nabla}_a \phi + \sum_{i=1,2,3} \delta^{\mathbf{i}} e_{\mathbf{i}}^a \hat{\nabla}_a \phi)^2}{\beta^2} - \sum_{i=1,2,3} (e_{\mathbf{i}}^a \hat{\nabla}_a \phi)^2$$

and the restriction to Σ_t of $\star\nabla_b\phi$ is:

$$\begin{aligned} i^\star_{\Sigma_t}(\star\nabla_b\phi) &= \frac{(f^a\hat{\nabla}_a\phi + \sum_{i=1,2,3}\delta^{\mathbf{i}}e^a_{\mathbf{i}}\hat{\nabla}_a\phi)}{\beta^2} i^\star_{\Sigma_t}(\star f_a) \\ &= \left(f^a\hat{\nabla}_a\phi + \sum_{i=1,2,3}\delta^{\mathbf{i}}e^a_{\mathbf{i}}\hat{\nabla}_a\phi\right)\frac{e^a_1\wedge e^a_2\wedge e^a_3}{\beta}. \end{aligned} \tag{2.18}$$

Using these results, the energy 3-form is given by:

$$i^\star_{\Sigma_t}(\star\hat{T}^aT_{ab}) = \hat{T}^a\hat{\nabla}_a\phi\left(\left(f^a\hat{\nabla}_a\phi + \sum_{i=1,2,3}\delta^{\mathbf{i}}e^a_{\mathbf{i}}\hat{\nabla}_a\phi\right)\frac{e^a_1\wedge e^a_2\wedge e^a_3}{\beta}\right)$$

$$+\left(-\frac{1}{2}\left(\frac{(f^a\hat{\nabla}_a\phi + \sum_{i=1,2,3}\delta^{\mathbf{i}}e^a_{\mathbf{i}}\hat{\nabla}_a\phi)^2}{\beta^2} - \sum_{i=1,2,3}(e^a_{\mathbf{i}}\hat{\nabla}_a\phi)^2\right) + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)||\hat{T}^a||\beta e^a_1\wedge e^a_2\wedge e^a_3$$

$$= \left\{\frac{1}{2}\left((f^c\hat{\nabla}_c\phi)^2 + \sum_{i=1,2,3}(e^a_i\hat{\nabla}_a\phi)^2(\beta^2-(\delta^{\mathbf{i}})^2)\right) + \beta^2\left(\frac{\phi^2}{2}+b\frac{\phi^4}{4}\right)\right\}\frac{||\hat{T}^a||^2}{\beta}e^a_1\wedge e^a_2\wedge e^a_3$$

To get the equivalence, it is sufficient to notice that:

- $\hat{g}_{cd}\hat{T}^c\hat{T}^d$ is a positive function over the compact $\overline{V}$ and, as such, is bounded below and behind by two positive constants;
- as already noticed in remark 2.20, β is a positive function over $\overline{V}$;
- and, finally, the scalar products β, $\delta_{\mathbf{i}}$ and the difference $\beta^2 - \delta_{\mathbf{i}}$ are clearly bounded below by 1 and above by a certain constant since we are working on a compact.

The energy equivalence then holds:

$$\int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^aT_{ab}\right) \approx \int_{\Sigma_t}\left(\frac{(f^c\hat{\nabla}_c\phi)}{2}^2 + \frac{1}{2}\sum_{i=1,2,3}(e^a_{\mathbf{i}}\hat{\nabla}_a\phi)^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)e^a_1\wedge e^a_2\wedge e^a_3 \blacklozenge$$

From now on, the strategy is exactly the same as in subsection 2.2.2. The fact that the energy over Σ dominates the energy on all slices Σ_t is established:

Proposition 2.23. *The following estimate holds, for all t in $[0,1]$:*

$$E(\Sigma_t) \lesssim E(\Sigma)$$

Proof. Let t be in $[0,1]$. The energy 3-form is integrated over the surfaces Σ, Σ_t and S^1_t; Stokes theorem then gives:

$$\begin{aligned} &\int_\Sigma i^\star_\Sigma\left(\star\hat{T}^aT_{ab}\right) - \int_{S^1_t} i^\star_{S_{u_0}}\left(\star\hat{T}^aT_{ab}\right) - \int_{\Sigma_t} i^\star_{\Sigma_t}\left(\star\hat{T}^aT_{ab}\right) \\ &= \int_{V^1_t}\left(\hat{\nabla}^{(a}\hat{T}^{b)}T_{ab} + (1-\frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi\hat{T}^a\hat{\nabla}_a\phi + \hat{T}^a\hat{\nabla}_a\frac{\phi^4}{4}\right)\mu[\hat{g}]. \end{aligned}$$

As in the proof of proposition 2.19, the error term is estimated considering that $\nabla^{(a}\hat{T}^{b)}$ has bounded coefficients in the given basis, that $\mathrm{Scal}_{\hat{g}}$ is bounded and using the behavior of b. The volume form is decomposed on the basis $(\mathrm{d}t, e_a^{\mathbf{i}})_{1,2,3}$ as:

$$\mu[\hat{g}] = \frac{\mathrm{d}t \wedge e_a^1 \wedge e_a^2 \wedge e_a^3}{e_0^a \hat{\nabla}_a t},$$

where $e_0^a \hat{\nabla}_a t$ is a positive function and, as a consequence, is bounded above and below by two positives constants.

We obtain, when contracting the volume form through e_0^a in order to integrate in time:

$$\left| \int_{V_t^1} \left(\hat{\nabla}^{(a}\hat{T}^{b)} T_{ab} + (1 - \frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi \hat{T}^a \hat{\nabla}_a \phi + \hat{T}^a \hat{\nabla}_a \frac{\phi^4}{4} \right) \mu[\hat{g}] \right| \lesssim \int_t^1 E(\Sigma_t)\mathrm{d}t.$$

Using the energy equivalence proved in proposition 2.21, the following inequality then holds:

$$E(S_t^1) + E(\Sigma_t) \lesssim \int_t^1 E(\Sigma_t)\mathrm{d}t + E(\Sigma). \tag{2.19}$$

Since $E(S_t^1)$ is non negative (see lemma 1.5 and proposition 2.3), (2.19) turns into the integral inequality:

$$E(\Sigma_t) \lesssim \int_t^1 E(\Sigma_t)\mathrm{d}t + E(\Sigma).$$

Using Growall's lemma, we get:

$$E(\Sigma_t) \lesssim E(\Sigma) \blacklozenge.$$

Finally, the following proposition holds:

Proposition 2.24. *The following estimates are satisfied on V:*

$$E(\Sigma) \approx E(S_{u_0}) + E(\Sigma_0^{u_0<})$$

Proof. The energy 3-form is integrated over the surfaces Σ_0, Σ_t and S_0^t; the application of Stokes theorem gives:

$$\begin{aligned} &\int_{\Sigma_t} i^\star_{\Sigma_t} \left(\star \hat{T}^a T_{ab} \right) - \int_{S_0^t} i^\star_{S_{u_0}} \left(\star \hat{T}^a T_{ab} \right) - \int_{\Sigma_0} i^\star_{\Sigma_0} \left(\star \hat{T}^a T_{ab} \right) \\ &= \int_{V_0^t} \left(\hat{\nabla}^{(a}\hat{T}^{b)} T_{ab} + (1 - \frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi \hat{T}^a \hat{\nabla}_a \phi + \hat{T}^a \hat{\nabla}_a \frac{\phi^4}{4} \right) \mu[\hat{g}]. \end{aligned}$$

The error term satisfy the same inequality as in proposition 2.23:

$$\left| \int_{V_0^t} \left(\hat{\nabla}^{(a}\hat{T}^{b)} T_{ab} + (1 - \frac{1}{6}\mathrm{Scal}_{\hat{g}})\phi \hat{T}^a \hat{\nabla}_a \phi + \hat{T}^a \hat{\nabla}_a \frac{\phi^4}{4} \right) \mu[\hat{g}] \right| \lesssim \int_0^t E(\Sigma_s)\mathrm{d}s.$$

As a consequence, the two following inequalities hold:

$$E(S_0^t) + E(\Sigma_0) \lesssim \int_0^t E(\Sigma_t)\mathrm{d}t + E(\Sigma_t) \tag{2.20}$$

$$E(\Sigma_t) \lesssim \int_0^t E(\Sigma_t)\mathrm{d}t + E(S_0^t) + E(\Sigma_0). \tag{2.21}$$

The right-hand side of inequality (2.20) is estimated via proposition 2.23 as:

$$\int_0^t E(\Sigma_t)\mathrm{d}t \lesssim E(\Sigma)$$

and, as a consequence, for $t = 1$, the first part of the equivalence can be stated:

$$E(S_{u_0}) + E(\Sigma_0) \lesssim E(\Sigma_1).$$

Using the positivity of $E(S_0^t)$, inequality (2.21) becomes:

$$E(\Sigma_t) \lesssim \int_0^t E(\Sigma_t)\mathrm{d}t + E(S_{u_0}) + E(\Sigma_0).$$

Using Gronwall's lemma and setting $t = 1$, the second part of the equivalence is obtained:

$$E(\Sigma) \lesssim E(S_{u_0}) + E(\Sigma_0).\blacklozenge$$

2.4 Final estimates

Finally, using the three propositions 2.8, 2.19 and 2.24, the following a priori global estimates hold:

Theorem 2.25. *Let u be a smooth solution with compactly supported data of the nonlinear wave equation:*

$$\Box u + \frac{1}{6}Scal_{\hat{g}}u + bu^3 = 0.$$

Then, the a priori estimates hold:

$$E(\Sigma_0) \approx E(\mathcal{I}^+).$$

Remark 2.26.

1. *The solution of the wave equation is assumed to be smooth in order to avoid the problem of defining trace operators for weak solutions of the equation. Nonetheless, using a usual trace theorem, as soon as u is in $H^{\frac{3}{2}}(\hat{M})$, its trace over $\mathcal{I}^+$ and Σ_0 is well defined.*

2. *Furthermore, in the framework of a (characteristic) Cauchy problem, it is know that the solution is in $H^1(\hat{M})$. Using the same theorem of existence of trace operators, u is only in $H^{\frac{1}{2}}(\Sigma_0)$ or $H^{\frac{1}{2}}(\mathcal{I}^+)$, which is clearly not sufficient to write such estimates. It will be shown in section 4.1 that these operators are well defined and with values in $H^1(\Sigma_0)$ or $H^1(\mathcal{I}^+)$.*

Proof. Let us consider the hypersurface Σ_0. This hypersurface is split in $\Sigma_0^{u_0<}$ and $\Sigma_0^{u_0>}$:

$$E(\Sigma_0) = E(\Sigma_0^{u_0<}) + E(\Sigma_0^{u_0>}).$$

Using proposition 2.8:

$$E(\Sigma_0^{u_0>}) \lesssim E(\mathcal{I}^+_{u_0}) + E(S_{u_0}),$$

proposition 2.24:

$$E(S_{u_0}) + E(\Sigma_0^{u_0<}) \lesssim E(\Sigma_T),$$

and proposition 2.19:

$$E(\Sigma_T) \lesssim E(\mathcal{I}^+_T),$$

we obtain the first part of the apriori estimate:

$$E(\Sigma_0) \lesssim E(\mathscr{I}^+_T) + E(\mathscr{I}^+_{u_0}) = E(\mathscr{I}^+).$$

Conversely, let us consider

$$E(\mathscr{I}^+) = E(\mathscr{I}^+_{u_0}) + E(\mathscr{I}^+_T).$$

Using proposition 2.19:

$$E(\mathscr{I}^+_T) \lesssim E(\Sigma_T),$$

proposition 2.24:

$$E(\Sigma_T) \lesssim E(S_{u_0}) + E(\Sigma_0^{u_0<}),$$

and proposition 2.8:

$$E(S_{u_0}) + E(\mathscr{I}^+_{u_0}) \lesssim E(\Sigma_0^{u_0>}),$$

we get the other side of the inequality:

$$E(\mathscr{I}^+) \lesssim E(\Sigma_0^{u_0>}) + E(\Sigma_0^{u_0<}) = E(\Sigma_0).\blacklozenge$$

3 Goursat problem on $\mathscr{I}^+$

We show in this section that there exists a unique solution to the Goursat problem on $\mathscr{I}^+$ for characteristic data in $H^1(\mathscr{I}^+)$:

$$\begin{cases} \hat{\Box}\phi + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\phi + b\phi^3 = 0 \\ \phi\big|_{\mathscr{I}^+} = \theta \in H^1(\mathscr{I}^+). \end{cases} \tag{3.1}$$

It is known (see [53]) that the linear Goursat problem on a smooth weakly hypersurface admits a global solution; nonetheless, due to technical problems coming from the singularity at i^0 (essentially the fact that some Sobolev embeddings are not valid), the existence of a global solution must be justified carefully.

The proof of the existence for problem (3.1) is made in three steps:

1. considering two solutions of the wave equation, estimates on the difference of these solutions are established; the main technical problem, which consists in obtaining uniform Sobolev estimates, is encountered and solved in section 3.1.

2. Let $\mathcal{S}$ be a uniformly spacelike hypersurface for the metric $\hat{g}$ in the future of Σ_0 and close enough to $\mathscr{I}^+$. The existence of solutions for problem (3.1) with characteristic data whose compact support contains neither i^0 nor i^+ is obtain through a Picard iteration in the future of $\mathcal{S}$ for small data in section 3.2.

3. Then this solution is extended down to Σ_0, by means of a Cauchy problem on $\mathcal{S}$ and density results in section 3.3.1.

4. Finally, using estimates for the propagator between $\mathscr{I}^+$ and Σ_0 obtained in section 3.1, the result is extended to $H^1(\mathscr{I}^+)$ for small data in section 3.3.2.

3.1 Continuity result

This section is devoted to the proof of a continuity result in function of the characteristic data, although an existence theorem has not yet been stated. Consequences of these estimates wil be required to obtain well-posedness of problem (3.1).

3.1.1 Technical tools

The greatest problem when dealing with foliations is the difficulty to obtain uniform estimates of the non-linearity. This requires that the constants associated with the Sobolev embeddings are controlled uniformly over the foliation. Two theorems are given to control these constants.

The first one is a result by Hébey-Vaugon which is used to obtain a uniform control over the Sobolev constant for the embedding from H^1 into L^6 on the leaves of a foliation. The result is the following (theorem 7.2 in [51], adapted to dimension 3) :

Theorem 3.1 (Hébey–Vaugon, 1995)**.** *Let (Σ, g) be a smooth complete Riemannian manifold of dimension 3. Suppose that its Riemann curvature R_{ijkl} and its injectivity radius inj_g satisfy:*

$$\exists(\lambda_1, \lambda_1, i), ||R_{ijkl}||_g \leq \lambda_1,\ ||\nabla^a R_{ijkl}||_g \leq \lambda_2 \text{ and } inj_g \geq i.$$

Then there exists a constant B depending only on λ_1, λ_2 and i such that, for all functions u in $H^1(M)$:

$$\left(\int_M |u|^6 d\mu[g]\right)^3 \leq \sqrt{\frac{4}{3\omega_3^{\frac{2}{3}}}} \int_M |\nabla u|^2 d\mu[g] + B \int_M u^2 d\mu[g],$$

where ω_3 is the volume of the unit sphere in $\mathbb{R}^4$.

The second useful result is an extension theorem. It is used in the following as an intermediary result to apply theorem 3.1. As previously, it is necessary to control the norm of the map (see chapter VI, theorems 5 and 5' in [81]):

Theorem 3.2 (Extension theorem)**.** *Let U be a bounded open set in $\mathbb{R}^n$; we assume that its boundary ∂U is C^1 differentiable.*
Then there exists an operator $\mathcal{E}$ from $H^1(U)$ into $H^1(\mathbb{R}^n)$ such that:

1. *for all f in $H^1(U)$, $\mathcal{E}(f)\Big|_U = f$; $\mathcal{E}$ is an extension operator;*
2. *there exists a constant C depending only on the Lipschitz constant of the boundary such that, for all f in $H^1(U)$:*

$$||\mathcal{E}(f)||_{H^1(\mathbb{R}^n)} \leq C||f||_{H^1(U)}.$$

Remark 3.3. *There exist variations of this result: the constant C can de replaced by the L^∞-norm of the maps on the boundary or the curvatures and its derivatives when ∂U is more regular. More precise estimates can be found in [9].*

3.1.2 Uniform Sobolev estimates on uniformly spacelike foliations

When dealing with apriori estimates in section 2, the problem of controlling the nonlinearity was avoided by assumptions on the asymptotic behavior of the function b. Controlling the non-linearity in function of the H^1-norm is a way to remove these assumptions. This is done by obtaining uniform Sobolev embeddings over a foliation.

Proposition 3.4. *Let (M, g) be a four dimensional smooth Lorentzian manifold; let $(\Sigma_t)_{t\in I}$ be a foliation of M by uniformly spacelike hypersurfaces with smooth boundary; we assume that I is a compact interval in $\mathbb{R}$ and that the hypersurfaces Σ_t can be embedded in simply connected open sets of $\mathbb{R}^3$.*

Then there exists a constant K_{sob}, depending only on the geometry of the foliation such that:

$$\forall t \in I, \forall f \in H^1(\Sigma_t), ||f||_{L^6(\Sigma_t)} \leq K_{Sob}||f||_{H^1(\Sigma_t)}.$$

Remark 3.5. *This constant depends on:*

- *the supremum of the Lipshitz constant of the boundaries of the hypersurfaces Σ_t;*
- *the supremum of the curvatures and its derivatives of the hypersurfaces of a given extension of the Σ_t;*
- *the infimum of the injectivity radii of a given extension of the Σ_t.*

Proof. To obtain such estimates, the method is the following:

1. we notice that the boundaries of the hypersurfaces Σ_t have the same Lipschitz norm L;
2. the slices Σ_t can be considered as a family of compact hypersurfaces with trivial topology which can be extended in $\mathbb{R}^3$; the metrics $g\big|_{\Sigma_t}$ are extended smoothly to $\mathbb{R}^3$ so that they are equal to the euclidean metric of $\mathbb{R}^3$ outside a compact set; these extensions are denoted by $\tilde{\Sigma}_t$ and their metrics $\tilde{g}_t$;
3. we then obtain a family of unbounded 3-dimensional manifolds $(\tilde{\Sigma}_t, \tilde{g}_t)$; since these manifolds are Euclidean outside a compact set, there exist three constants λ_1, λ_2, i such that, uniformly in t:
$$\exists(\lambda_1, \lambda_1, i), ||R_{ijkl}|| \leq \lambda_1, \; ||\nabla^a R_{ijkl}|| \leq \lambda_2 \text{ and } inj_{\tilde{g}_t} \geq i.$$
4. using theorem 3.1, there exists a constant such that uniformly in t the following Sobolev embeddings hold:
$$\forall t \in I, \forall f \in H^1(\tilde{\Sigma}_t), ||f||_{L^6\tilde{\Sigma}_t} \leq K_1||f||_{H^1\tilde{\Sigma}_t}$$
5. using theorem 3.2, there exists a family of extension operators $\mathcal{E}_t$ from $H^1(\Sigma_t)$ into $H^1(\tilde{\Sigma}_t)$. Since the boundaries of Σ_t have the same Lipschitz constant L, there exists a constant K_2, which depends only on L, such that
$$\forall t \in I, \forall f \in H^1(\Sigma_t), \; ||\mathcal{E}_t(f)||_{H^1(\tilde{\Sigma}_t)} \leq K_2||f||_{H^1(\tilde{\Sigma}_t)}$$

Finally using these extensions and the Sobolev imbeddings from $H^1(\tilde{\Sigma}_t)$ into $L^6(\tilde{\Sigma}_t)$, we obtain, for all n and t:

$$\begin{aligned} ||u_n||_{L^6(\Sigma_t)} &\leq ||\mathcal{E}_t(u_n)||_{L^6(\tilde{\Sigma}_t)} \\ &\leq K_1||\mathcal{E}_t(u_n)||_{H^1(\tilde{\Sigma}_t)} \\ &\leq K_2K_1||u_n||_{H^1(\Sigma_t)}. \end{aligned}$$

We denote by K_{Sob} the constant K_1K_2.♦

3.1.3 Continuity in terms of the initial data for the Cauchy problem

The purpose of this section is to establish estimates on the difference of two solutions of the wave equation. The purpose of these estimates is to obtain continuity in terms of initial data, characteristic or not. This step is an important one to obtain the continuity of the scattering operator.

Finally, it is important to notice that the proof which is made here does not require that the function b vanishes on $\mathcal{I}^+$ or that it satisfies the decay condition:

$$|\hat{T}^a\hat{\nabla}_a b| \leq cb.$$

Using the method developed below, it is then possible to obtain the apriori estimates of section 2.

Let ϕ and ψ be two smooth functions:

$$\hat{\Box}u + \frac{1}{6}\mathrm{Scal}_{\hat{g}}u + bu^3 = 0.$$

We assume that they satisfy one of the problems:

- an initial value problem on Σ_0 with data in $H^1_0(\Sigma_0) \times L^2(\Sigma_0)$ with compact support in Σ_0;
- an characteristic initial value problem with data in $H^1(\mathcal{I}^+)$ with compact support which contains neither i^0 nor i^+.

This ensures that the support of u and v does not contain the singularity i^0.

Theorem 3.6. *Let ϕ and ψ two smooth solutions of the nonlinear problem:*

$$\Box u + \frac{1}{6}Scal_{\hat{g}}u + bu^3 = 0.$$

Then there exist two constants, depending on $Scal_{\hat{g}}$, $\hat{\nabla}^{(a}\hat{T}^{b)}$, b and the energies of ϕ and ψ on Σ_0 and $\mathcal{I}^+$ such that the following estimates hold:

$$||\phi - \psi||^2_{H^1(\mathcal{I}^+)} \leq C(E_\phi(\Sigma_0), E_\psi(\Sigma_0))E_{\phi-\psi}(\Sigma_0)$$

and

$$E_{\phi-\psi}(\Sigma_0) \leq \tilde{C}(||\phi||_{H^1(\mathcal{I}^+)}, ||\psi||_{H^1(\mathcal{I}^+)})||\phi - \psi||^2_{H^1(\mathcal{I}^+)},$$

where the energy of a function on Σ_0 is chosen to be:

$$E_\phi(\Sigma_0) \approx \int_{\Sigma_0} i^\star_{\Sigma_0}\left(\hat{T}^a T_{ab}\right) \approx \int_{\Sigma_0} (\hat{T}^a\hat{\nabla}\phi)^2 + \sum_{i=1,2,3}(e^a_i\hat{\nabla}_a\phi)^2 + \phi^2\, d\mu_{\Sigma_0},$$

where T_{ab} is the energy tensor associated with the linear wave equation and $(e^a_i)_{i=1,2,3}$ is an orthonormal basis of $T\Sigma_0$.

Remark 3.7. *Because of the a priori estimates, the constants can be chosen indifferently to depend on the energy on $\mathcal{I}^+$ or Σ_0.*

Proof. The proof relies on exactly the same strategy as in the first section when establishing the a priori estimates. Let δ be the difference between ϕ and ψ:

$$\delta = \phi - \psi.$$

δ satisfies the partial differential equation:

$$\Box\delta + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\delta + b(\psi^2 + \psi\phi + \phi^2)\delta = 0.$$

To establish the inequality, let us consider the energy tensor associated with the linear equation:

$$T_{ab} = \hat{\nabla}_a\delta\hat{\nabla}_b\delta + \hat{g}_{ab}\left(-\frac{1}{2}\hat{\nabla}_c\delta\hat{\nabla}^c\delta + \frac{\delta^2}{2}\right).$$

The error term associated with this tensor is:

$$\hat{\nabla}^a(\hat{T}^b T_{ab}) = (\hat{\nabla}^{(a}\hat{T}^{b)}T_{ab}) + \underbrace{\delta(\hat{T}^a\hat{\nabla}_a\delta) - (\hat{T}^a\hat{\nabla}_a\delta)\left(\frac{1}{6}\mathrm{Scal}_{\hat{g}}\delta + b(\psi^2 + \psi\phi + \phi^2)\delta\right)}_{A}.$$

The term A can be estimated by:

$$\begin{aligned} A &\leq \frac{1}{2}\left(\delta^2 + (\hat{T}^a\hat{\nabla}_a\delta)^2\right) + \frac{1}{2}\sup_{\hat{M}}\left(|\mathrm{Scal}_{\hat{g}}|\right)\left(\delta^2 + (\hat{T}^a\hat{\nabla}_a\delta)^2\right) + 2\sup_{\hat{M}}|b|\left((\hat{T}^a\hat{\nabla}_a\delta(\phi^2 + \psi^2)\delta)\right) \\ A &\leq 2\max\left(\sup_{\hat{M}}\left(|\mathrm{Scal}_{\hat{g}}|\right), \sup_{\hat{M}}|b|, 1\right)\left((\hat{T}^a\hat{\nabla}_a\delta)^2 + \delta^2 + \delta^2\psi^4 + \delta^2\phi^4\right). \end{aligned}$$

The estimates will then be obtained in exactly the same way as in section 1, provided that we are able to use the Sobolev embeddings on the spacelike slices. The main problem then arises when working in the Schwarzschild section because of the choice of the foliation $\mathcal{H}_s$ which contains the singularity i^0.

On U, the analogue of the equation (2.17) is:

$$\begin{aligned} &E_\delta(\Sigma_t) - ||\delta||^2_{H^1(\mathcal{I}^+_T)} \\ &\leq \max\left\{1, \sup\left(|\mathrm{Scal}_{\hat{g}}|\right), \sup\left(|b|\right), \sup\left(||\nabla^{(a}\hat{T}^{b)}||\right)\right\}\left(\int_t^{T_{max}}\left(E_\delta(\Sigma_t) + \int_{\Sigma_t}\left(\delta^2\psi^4 + \delta^2\phi^4\right)\mu_{\Sigma_t}\right)\mathrm{d}t\right), \end{aligned}$$

where:

$$E_\delta(\Sigma_t) = \int_{\Sigma_t}\frac{1}{2}\left(\sum_{i=0}^{4}(e^a_{\mathbf{i}}\nabla_a\delta)^2 + \frac{\delta^2}{2}\right)\mu_{\Sigma_t} = \frac{1}{2}\left(||u||^2_{H^1(\Sigma_t)} + ||\hat{T}^a\hat{\nabla}_a\delta||^2_{L^2(\Sigma_t)}\right). \quad (3.2)$$

The foliation (Σ_t) satisfies the assumption of proposition 3.4: using this proposition, the integral inequality then becomes:

$$\begin{aligned} &E_\delta(\Sigma_t) - ||\delta||^2_{H^1(\mathcal{I}^+_T)} \\ &\leq \left(C_1 + ||\phi||^4_{H^1(\Sigma)} + ||\psi||^4_{H^1(\Sigma)}\right)\left(\int_t^{T_{max}} E_\delta(\Sigma_t)\mathrm{d}t\right). \end{aligned}$$

This gives, using Gronwall estimates:

$$E_\delta(\Sigma_t) \leq \exp\left(\left(C_1 + ||\phi||^4_{H^1(\Sigma)} + ||\psi||^4_{H^1(\Sigma)}\right)(T_{max} - t)\right) ||\delta||^2_{H^1(\mathcal{I}^+_T)}.$$

The other estimate is obtained when noticing that the inequality also holds:

$$\begin{aligned} &||\delta||^2_{H^1(\mathcal{I}^+_T)} - E_\delta(\Sigma_t) \\ \leq &\left(C_1 + ||\phi||^4_{H^1(\Sigma)} + ||\psi||^4_{H^1(\Sigma)}\right)\left(\int_t^{T_{max}} E_\delta(\Sigma_t)\mathrm{d}t\right). \end{aligned}$$

Using proposition 2.19, the H^1-norm of ϕ and ψ on Σ is controlled by the H^1-norm of ψ and ϕ on $\mathcal{I}^+_T$ and, as a consequence, by the H^1-norm of ψ and ϕ on $\mathcal{I}^+$. We then finally obtain on U the following inequality: there exist two increasing functions c_U and C_U such that:

$$\begin{aligned} E_\delta(\Sigma) &\leq c_U\left((||\phi||^4_{H^1(\mathcal{I}^+)} + ||\psi||^4_{H^1(\mathcal{I}^+)}\right) ||\delta||^2_{H^1(\mathcal{I}^+)} \\ ||\delta||^2_{H^1(\mathcal{I}^+)} &\leq C_U\left((||\phi||^4_{H^1(\mathcal{I}^+)} + ||\psi||^4_{H^1(\mathcal{I}^+)}\right) E_\delta(\Sigma) \end{aligned} \tag{3.3}$$

On V, the principle is exactly the same: the only modification comes from the fact that a new boundary term arises, corresponding to the boundary of the Schwarzschild section. We work with the same geometric configuration. The equivalent of equation (2.19) is then:

$$E_\delta(S^1_t) + E_\delta(\Sigma_t) - E_\delta(\Sigma) \leq C_2 \int_t^1 E_\delta(\Sigma_t)\mathrm{d}t + \int_{\Sigma_t} \delta^2\psi^4 + \delta^2\phi^4 \mathrm{d}t$$

where E_δ has the same expression as in equation (3.2). We finally obtain the energy equivalence:

$$\begin{aligned} E_\delta(\Sigma) &\leq c_V\left((||\phi||^4_{H^1(\mathcal{I}^+)} + ||\psi||^4_{H^1(\mathcal{I}^+)}\right)\left(E_\delta(\Sigma_0^{u_0<}) + E_\delta(S_{u_0})\right) \\ E_\delta(\Sigma_0^{u_0<}) + E_\delta(S_{u_0}) &\leq C_V\left((||\phi||^4_{H^1(\mathcal{I}^+)} + ||\psi||^4_{H^1(\mathcal{I}^+)}\right) E_\delta(\Sigma) \end{aligned} \tag{3.4}$$

Remark 3.8. *In the subset V of $\hat{M}$, the energy on a slice is controlled by the upper slice, which is denoted by Σ as said in proposition 2.23. As this energy is controlled by proposition 2.19 by the H^1-norm on $\mathcal{I}^+_T$ and, as a consequence, on $\mathcal{I}^+$, this explains why the energy on $\mathcal{I}^+$ appears in the inequality.*

Finally, on $\Omega^+_{u_0}$, it is not possible to use the same method as above to control uniformly the Sobolev constant. The strategy consists in adopting the same foliation by the hypersurfaces $\mathcal{H}_s$. The energy on this foliation is weighted Sobolev norm with a precise decay. The Sobolev embeddings must then be adaptated to that decay. The identifying vector field is used to write the integral (see formulae (2.9) and (2.10)). The error term can then be expressed as:

$$\begin{aligned} \int_0^{\tau(s)} \Bigg(\int_{\mathcal{H}_\tau} \Big\{ &4mR^2(3+uR)(\partial_R\phi)^2 + (1-12mR)\phi\left(u^2\partial_u\phi - 2(1+uR)\partial_R\phi\right) \\ &-(\hat{T}^a\hat{\nabla}_a\delta)\delta(\phi^2+\phi\psi+\psi^2)\Big\}(r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{\frac{R}{|u|}}\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}\Bigg)\mathrm{d}\tau \end{aligned}$$

In this subset of $\hat{M}$, the error is bounded above by, using Hölder estimates:

$$\int_0^{\tau(s)} \left\{ E_\delta(\mathcal{H}_\tau) + \left(\int_{\mathcal{H}_\tau} \delta^6 \sqrt{\frac{R}{|u|}} \mathrm{d}u \wedge \mathrm{d}\omega_{\mathbb{S}^2} \right)^{\frac{1}{3}} \left(\int_{\mathcal{H}_\tau} \left(\phi^6 + \psi^6\right) \sqrt{\frac{R}{|u|}} \mathrm{d}u \wedge \mathrm{d}\omega_{\mathbb{S}^2} \right)^{\frac{2}{3}} \right\} \mathrm{d}\tau$$
$$\leq \int_0^{\tau(s)} \left\{ E_\delta(\mathcal{H}_\tau) + \sqrt{\frac{\epsilon}{2m|u_0|}} \left(\int_{\mathcal{H}_\tau} \delta^6 \mathrm{d}u \wedge \mathrm{d}\omega_{\mathbb{S}^2} \right)^{\frac{1}{3}} \left(\int_{\mathcal{H}_\tau} \left(\phi^6 + \psi^6\right) \mathrm{d}u \wedge \mathrm{d}\omega_{\mathbb{S}^2} \right)^{\frac{2}{3}} \right\} \mathrm{d}\tau.$$

The Sobolev embedding from H^1 into L^6 must then be realized uniformly in τ with regard to the volume form $\mathrm{d}u \wedge \omega_{\mathbb{S}^2}$, which is the volume form associated with the cylinder $]u_0, +\infty[\times\mathbb{S}^2$ and the metric $(\mathrm{d}u)^2 + \mathrm{d}\omega_{\mathbb{S}^2}^2$. Since the Sobolev embedding from $H^1(]u_0, +\infty[\times\mathbb{S}^2)$ into $L^6(]u_0, +\infty[\times\mathbb{S}^2)$ is valid in this geometry, we obtain, in the coordinate system $(u, \omega_{\mathbb{S}^2})$ the following Sobolev inequality:

$$\left(\int_{\mathcal{H}_s} \phi^6 \mathrm{d}u \mathrm{d}\omega_{\mathbb{S}^2} \right)^{\frac{1}{3}} = \left(\int_{]u_0,+\infty[\times\mathbb{S}^2} \phi^6 \mathrm{d}u \mathrm{d}\omega_{\mathbb{S}^2} \right)^{\frac{1}{3}}$$
$$\leq K \int_{]u_0,+\infty[\times\mathbb{S}^2} (\partial_u (\phi\big|_{\mathcal{H}_s = \{u=-sr^*\}}))^2 + |\nabla_{\mathbb{S}^2}\phi|^2 + \phi^2 \mathrm{d}u \mathrm{d}\omega_{\mathbb{S}^2}$$
$$\leq \int_{]u_0,+\infty[\times\mathbb{S}^2} \left(\partial_u \phi + \frac{r^* R^2 (1-2mR)}{|u|} \partial_R \phi \right)^2 + |\nabla_{\mathbb{S}^2}\phi|^2 + \phi^2 \mathrm{d}u \mathrm{d}\omega_{\mathbb{S}^2}$$
$$\leq \int_{]u_0,+\infty[\times\mathbb{S}^2} 2(\partial_u\phi)^2 + 2(r^*R)^2(1-2mR)^2 \left(\frac{R}{|u|}\right)^2 (\partial_R \phi^2)| + \nabla_{\mathbb{S}^2}\phi|^2 + \phi^2 \mathrm{d}u \mathrm{d}\omega_{\mathbb{S}^2}$$
$$\leq \int_{]u_0,+\infty[\times\mathbb{S}^2} 2\frac{u^2}{u_0^2}(\partial_u\phi)^2 + (1+\epsilon)^2 \frac{\epsilon}{2m|u_0|} \frac{R}{|u|} (\partial_R \phi^2) + |\nabla_{\mathbb{S}^2}\phi|^2 + \phi^2 \mathrm{d}u \mathrm{d}\omega_{\mathbb{S}^2}.$$

Then there exists a constant K, depending on u_0 and ϵ such that, uniformly in s:

$$\left(\int_{\mathcal{H}_\tau} \phi^6 \sqrt{\frac{R}{|u|}} \mathrm{d}u \mathrm{d}\omega_{\mathbb{S}^2} \right)^{\frac{2}{3}} \leq K ||\phi||^4_{H^1(\mathcal{H}_\tau)} \text{ and } \left(\int_{\mathcal{H}_\tau} \delta^6 \sqrt{\frac{R}{|u|}} \mathrm{d}u \mathrm{d}\omega_{\mathbb{S}^2} \right)^{\frac{1}{3}} \leq K E_\delta(\mathcal{H}_\tau).$$

Using the fact that (see equation (2.12)):

$$||\phi||^4_{H^1(\mathcal{H}_\tau(s))} \lesssim \left(E_\phi(\Sigma_0^{u_0>}) \right)^2,$$

the following integral inequality holds:

$$\left| E_\delta(\mathcal{H}_{\tau(s)} + E_\delta(S_{u_0}) - E_\delta(\Sigma_0^{u_0>}) \right|$$
$$\leq (C + K(E_\phi(\Sigma_0^{u_0>})^2 + E_\psi(\Sigma_0^{u_0>})^2)) \int_0^{\tau(s)} E_\delta(\mathcal{H}_\tau) \mathrm{d}\tau$$
$$\leq (C + K(E_\phi(\Sigma_0)^2 + E_\psi(\Sigma_0)^2)) \int_0^{\tau(s)} E_\delta(\mathcal{H}_\tau) \mathrm{d}\tau$$

and using the a priori estimates given by theorem 2.25,

$$\left| E_\delta(\mathcal{H}_{\tau(s)} + E_\delta(S_{u_0}) - E_\delta(\Sigma_0^{u_0>}) \right| \leq \left(C + \tilde{K} \left((||\phi||^4_{H^1(\mathscr{I}^+)} + ||\psi||^4_{H^1(\mathscr{I}^+)} \right) \right) \int_0^{\tau(s)} E_\delta(\mathcal{H}_\tau) \mathrm{d}\tau.$$

At the end, using the method as previously, there exist two increasing functions $c_{\Omega^+_{u_0}}$ and $C_{\Omega^+_{u_0}}$ such that:

$$\begin{aligned} \mathbb{E}_\delta(\mathcal{I}^+_{u_0}) + E_\delta(S_{u_0}) &\leq c_{\Omega^+_{u_0}}\left((||\phi||^4_{H^1(\mathcal{I}^+)} + ||\psi||^4_{H^1(\mathcal{I}^+)}\right) E_\delta(\Sigma_0^{u_0>}) \\ E_\delta(\Sigma_0^{u_0>}) &\leq C_{\Omega^+_{u_0}}\left((||\phi||^4_{H^1(\mathcal{I}^+)} + ||\psi||^4_{H^1(\mathcal{I}^+)}\right)\left(E_\delta(\mathcal{I}^+_{u_0}) + E_\delta(S_{u_0})\right). \quad (3.5) \end{aligned}$$

Eventually, combining the inequalities (3.3), (3.4) and (3.5) as in section 2 for the proof of theorem 2.25, we get the existence of two increasing functions c and C such that:

$$\begin{aligned} E_\delta(\Sigma_0) &\leq c\left((||\phi||^4_{H^1(\mathcal{I}^+)} + ||\psi||^4_{H^1(\mathcal{I}^+)}\right) ||\delta||^2_{H^1(\mathcal{I}^+)} \\ ||\delta||^2_{H^1(\mathcal{I}^+)} &\leq C\left((||\phi||^4_{H^1(\mathcal{I}^+)} + ||\psi||^4_{H^1(\mathcal{I}^+)}\right) E_\delta(\Sigma_0). \end{aligned}$$

Because of the a priori estimates given by theorem 2.25, the H^1-norm of ϕ and ψ on $\mathcal{I}^+$ can be replaced by the energy of ϕ and ψ on Σ_0.♦

This result is equivalent to the result obtained by Hörmander at the beginning of his paper.

Finally, as already mentioned, a by-product of this result is the continuity result for the Cauchy problem for the nonlinear wave equation on a uniformly spacelike hypersurface $\mathcal{S}$, transverse to $\mathcal{I}^+$. The problems are exactly the same: obtaining uniform Sobolev estimates near i^0 and in the equivalent of the region V. The techniques to solve this problem are then exactly the same. We are working with functions ϕ and ψ which satisfy the same assumptions as for theorem 3.6.

Proposition 3.9. *Let $\mathcal{S}$ be a uniformly spacelike hypersurface transverse to $\mathcal{I}^+$. Let ϕ and ψ be two smooth solutions of the nonlinear problem:*

$$\Box u + \frac{1}{6} Scal_{\hat{g}} u + bu^3 = 0.$$

Then there exists two constants, depending on $Scal_{\hat{g}}$, $\hat{\nabla}^a \hat{T}^b$, b and the energy of ϕ and ψ on Σ_0 and $\mathcal{S}$ such that the following estimates hold:

$$E_{\phi-\psi}(\mathcal{S}) \leq C(E_\phi(\Sigma_0), E_\psi(\Sigma_0)) E_{\phi-\psi}(\Sigma_0)$$

and

$$E_{\phi-\psi}(\Sigma_0) \leq \tilde{C}(||\phi||_{H^1(\mathcal{I}^+)}, ||\psi||_{H^1(\mathcal{I}^+)}) E_{\phi-\psi}(\mathcal{S}),$$

where the energy of a function u on a uniformly spacelike hypersurface Σ is chosen to be:

$$E_u(\Sigma) \approx \int_{\Sigma_0} (\hat{T}^a T_{ab}) \approx \int_{\Sigma_0} (\hat{T}^a \hat{\nabla} u)^2 + \sum_{i=1,2,3} (e^a_{\boldsymbol{i}} \hat{\nabla}_a u)^2 + u^2 d\mu_{\Sigma_0},$$

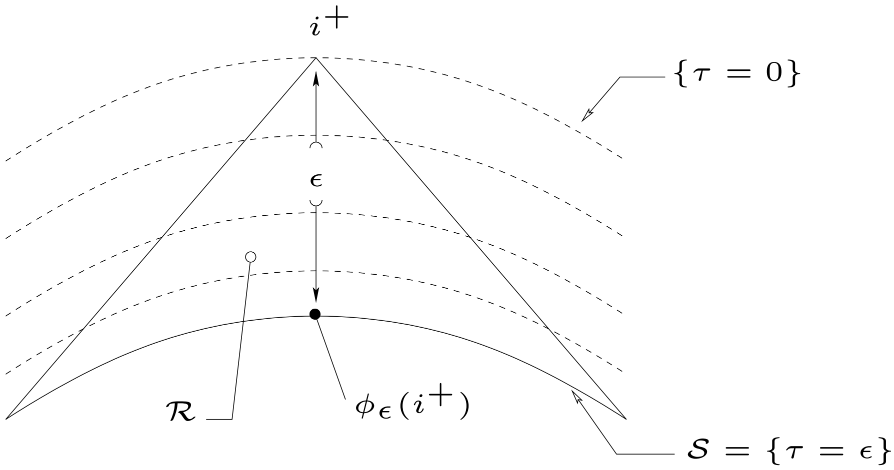
where T_{ab} is the energy tensor associated with the linear wave equation and $(e^a_{\boldsymbol{i}})_{i=1,2,3}$ is an orthonormal basis of $T\Sigma$.

3.2 Solution of the Goursat problem near $\mathcal{I}^+$ with small initial data.

The existence of solutions to the nonlinear problem is established from the linear problem through a Picard iteration in the future of a spacelike hypersurface $\mathcal{S}$, close enough to $\mathcal{I}^+$. This hypersurface is constructed as follows.

We choose to work here with a function in $H^1(\mathcal{I}^+)$ with compact support which contains neither i^0 nor i^+

Let τ be a "reverse" time function on $\hat{M}$, in the sense that its gradient is past directed with respect to $\hat{T}^a$. We assume that $\tau(i^+) = 0$. $\hat{M}$ is endowed with an orthonormal basis $(e^a_{\mathbf{i}})_{i=0,1,2,3}$ such that e^a_0 is colinear to $\hat{\nabla}\tau$ and $(e^a_{\mathbf{i}})_{i=1,2,3}$ is tangent to the time slices $\{\tau = constant\}$. The integral flow associated with ∂_τ is denoted by Φ_τ.



Let ϵ be a positive constant smaller than $\frac{1}{16}$ (see inequality (3.11))and consider $\Phi_\epsilon(i^+)$. Let $\mathcal{S}$ be a uniformly spacelike hypersurface for the metric $\hat{g}$ between $\mathcal{I}^+$ and $\{\Phi_\epsilon(p)|p \in \mathcal{I}^+\}$. We assume that $\mathcal{S}$ is uniformly spacelike, transverse to $\mathcal{I}^+$ in the past of the support of the characteristic data, θ, and contains $\Phi_\epsilon(i^+)$.

Remark 3.10. *The geometric framework is then exactly the same as in section 2.3: the timelike vector field which will be used for the energy is not colinear to the gradient defining the foliation. Nonetheless, since the hypersurface $\mathcal{S}$ is uniformly spacelike, the same estimates as in section 2.3 hold without the nonlinearity.*

Finally, the future of S in $\hat{M}$ is foliated by the surfaces $\mathcal{S}_\tau = \{\Phi_{\epsilon-\tau}(p)|p \in \mathcal{S}\}$ for τ in $[0,\epsilon]$ so that $\mathcal{S} = \mathcal{S}_\epsilon$ and $\mathcal{S}_0 = \{i^+\}$. The future of $\mathcal{S}$ is denoted by $\mathcal{R}$ and the subset of $\hat{M}$ between $\mathcal{S}_0$ and $\mathcal{S}_\tau$, $\mathcal{R}_\tau$.

The solution of the nonlinear problem is approximated via solutions of the linear problem on $\mathcal{I}^+$. Hörmander solved this problem in [53]:

Proposition 3.11 (Hörmander)**.** *Let us consider the linear inhomogeneous characteristic Cauchy problem on $\mathcal{I}^+$:*

$$\left\{ \begin{array}{l} \hat{\Box}\phi + \frac{1}{6}Scal_{\hat{g}}\phi = f \\ \phi\big|_{\mathcal{I}^+} = \theta \in H^1(\mathcal{I}^+). \end{array} \right.$$

where θ is a function whose compact support does not contain i^+ or i^0. Then, this problem admits a unique global solution in the future of Σ_0 in $C^0([0,\epsilon], H^1(\mathcal{S}_\tau))$.

Using this proposition and estimates for the linear problem, the following theorem holds:

Theorem 3.12. *Let us consider the nonlinear characteristic Cauchy problem on $\mathcal{I}^+$:*

$$\begin{cases} \hat{\Box} u + \frac{1}{6} Scal_{\hat{g}} u + bu^3 = 0 \\ u\big|_{\mathcal{I}^+} = \theta \in H^1(\mathcal{I}^+). \end{cases}$$

where θ is a function whose compact support does not contain i^+ or i^0. Then, for $||\theta||_{H^1(\mathcal{I}^+)}$ small enough, there exists a uniformly spacelike hypersurface $\mathcal{S}$ close enough to $\mathcal{I}^+$ such that this problem admits a smooth global solution on $\mathcal{R}$ in $C^0([0,\epsilon], H^1(\mathcal{S}_\tau))$.

Remark 3.13. *The proof of the well-posedness in $C^0([0,\epsilon], H^1(\mathcal{S}_\tau))$ is given in section 3.1.3 were the geometric estimates required to obtain it are established (see theorem 3.6 which remains true in that context).*

Proof. Let u_0 be a solution on $\mathcal{R}$ of the problem:

$$\begin{cases} \hat{\Box} \phi + \frac{1}{6} \mathrm{Scal}_{\hat{g}} \phi = 0 \\ \phi\big|_{\mathcal{I}^+} = \theta \in H^1(\mathcal{I}^+). \end{cases}$$

and let $(u_n)_{n\in\mathbb{N}}$ be the sequence of smooth functions on $\mathcal{R}$ defined by the recursion:

$$\begin{cases} \hat{\Box} u_{n+1} + \frac{1}{6} \mathrm{Scal}_{\hat{g}} u_{n+1} + bu_n^3 = 0 \\ u_{n+1}\big|_{\mathcal{I}^+} = \theta \in H^1(\mathcal{I}^+). \end{cases}$$

The sequence defined by the difference of two consecutive terms of this sequence is denoted by $(\delta_n = u_{n+1} - u_n)_{n\in\mathbb{N}}$. For $n \in \mathbb{N}$, the smooth function δ_n satisfies the Cauchy problem:

$$\begin{cases} \hat{\Box} \delta_n + \frac{1}{6} \mathrm{Scal}_{\hat{g}} \delta_n = -b(u_n^2 + u_n u_{n-1} + u_{n-1}^2)\delta_{n-1} \\ \delta_n = 0 \text{ on } \mathcal{I}^+. \end{cases}$$

The proof of the convergence is made in two steps: the first one consists in proving that, for initial data which are small enough, the sequence (u_n) is bounded; the second part proves the convergence of (u_n) by showing that the sequence (δ_n) is summable.

Proposition 3.14. *For $||\theta||_{H^1(\mathcal{I}^+)}$ small enough, the sequence $\sup_{\tau\in[0,\epsilon]} ||u_n||_{H^1(\mathcal{S}_\tau)}$ is bounded.*

Proof. Let n be a integer greater than 1. Let us finally consider the energy tensor associated with the linear wave equation:

$$T_{ab} = \hat{\nabla}_a u_{n+1} \hat{\nabla}_b u_{n+1} + \hat{g}_{ab} \left(-\frac{1}{2} \nabla_c u_{n+1} \nabla^c u_{n+1} + \frac{u_{n+1}^2}{2} \right).$$

The energy associated with a time slice $\mathcal{S}_\tau$ is written as:

$$E_{u_n}(\mathcal{S}_\tau) = \int_{\mathcal{S}_\tau} \left(\sum_{i=0,1,2,3} (e_{\mathbf{i}}^a \hat{\nabla} u_n)^2 + \frac{u_n^2}{2} \right) \mathrm{d}\mu_{\mathcal{S}_\tau}$$

and it is equivalent to $\int_{\mathcal{S}_\tau} i^\star(\star e_0^a T_{ab})$ (with constants which only depend on the geometric data, the L^∞-norm of b and the Killing form of e_0^a).

The error term associated to this energy tensor is:

$$\hat{\nabla}^a \left(e_0^b T_{ab} \right) = \hat{\nabla}^{(a} e_0^{b)} T_{ab} + u_{n+1} e_0^a \nabla_a u_{n+1} - \frac{1}{6} \mathrm{Scal}_{\hat{g}} u_{n+1} e_0^a \nabla_a u_{n+1} - (e_0^a \nabla_a u_{n+1}) b u_n^3,$$

which is smaller than, in absolute value:

$$|\hat{\nabla}^a e_0^b T_{ab}| \leq C \left(\sum_{i=0}^{3} (e_{\mathbf{i}}^a \hat{\nabla}_a u_{n+1})^2 + u_{n+1}^2 \right) + C u_n^6,$$

where C is a positive constant depending on $\sup(|\mathrm{Scal}_{\hat{g}}|)$, $\sup\left(||\hat{\nabla}^{(a} e_0^{b)}|| \right)$, $\sup(|b|)$ and the foliation $\mathcal{S}_\tau$.

The next step consists in using the Sobolev embedding of $H^1(\mathcal{S}_\tau)$ in $L^6(\mathcal{S}_\tau)$. There exist two obstacles to the use of this embedding:

1. the first is the fact that the estimates must be uniform over the foliation in the sense that it must not depend on the parameter τ of the foliation (or the Sobolev constant must be the same all along the foliation);

2. the second comes the fact that we must deal with the singularity in i^+.

To deal with the second problem, the manifold $\hat{M}$ is extended beyond $\mathcal{I}^+$ by pulling backwards the hypersurface $\mathcal{S}$ through the flow associated with the vector field ∂_τ of the time function τ. Since the regularity of the metric is arbitrarily smooth at i^+ (say, at least C^2, in order to insure the existence of the different curvatures), this gives an extension as a smooth Lorentzian manifold of the manifold $(\hat{M}, \hat{g})$ in the neighborhood of i^+. $\mathcal{I}^+$ is then the past light cone from i^+ obtained from a C^2-extension of the metric $\hat{g}$ behind i^+.

To obtain a Sobolev embedding from $H^1(\mathcal{S}_\tau)$ into $L^6(\mathcal{S}_\tau)$ uniformly in τ, it is necessary to have a uniform bound for the Sobolev constant on the hypersurface $\mathcal{S}_\tau$. This is achieved by using proposition 3.4 for the foliation $\mathcal{S}_\tau$: there exists a constant K_{Sob} depending on the foliation $\mathcal{S}_\tau$ such that, uniformly in τ:

$$\forall \tau \in [0, \epsilon], \forall u \in H^1(\mathcal{S}_\tau), ||u||_{L^6(\mathcal{S}_\tau)} \leq K_{Sob} ||u||_{H^1(\mathcal{S}_\tau)}.$$

Remark 3.15. *1. The constant K_{Sob} depends on the foliation and, as such, of the parameter ϵ.*

2. The hypersurfaces $\mathcal{S}_\tau$ shrink as τ tends to zero. This does not affect the fact that the Lipschitz constants for the $\mathcal{S}_\tau$, for $\tau > 0$ remain bounded. Furthermore, the initial data are taken to be with compact support away from i^0. As a consequence, the functions (u_n) vanish in a neighborhood of i^+.

A direct consequence of the uniform Sobolev embeddings of $H^1(\mathcal{S}_\tau)$ in $L^6(S_\tau)$ in dimension 3 is the following inequality:

$$\begin{aligned} \int_{\mathcal{S}_\tau} |\hat{\nabla}^a e_0^b T_{ab}| \mu_{\mathcal{S}_\tau} &\leq C E_{u_{n+1}}(\mathcal{S}_\tau) + C K_{sob} ||u_n||^6_{H^1(\mathcal{S}_\tau)} \\ &\leq C E_{u_{n+1}}(\mathcal{S}_\tau) + C K_{sob} \left(\sup_{\tau \in [0,\epsilon]} E_{u_n}(\mathcal{S}_\tau) \right)^3 . \end{aligned}$$

As a consequence, there exists a constant $\tilde{C}$ which depends only on the scalar curvature, the Killing form of e_0^a, the supremum of b and Sobolev constants such that:

$$E_{u_{n+1}}(\mathcal{S}_\tau) \leq \tilde{C}\left(\int_0^\tau E_{u_{n+1}}(\mathcal{S}_r)\mathrm{d}r + ||\theta||^2_{H^1(\mathcal{I}^+)} + \left(\sup_{r\in[0,\epsilon]}\left(E_{u_n}(\mathcal{S}_r)\right)\right)^3\right).$$

Remark 3.16. *The constant $\tilde{C}$ can be chosen arbitrarily high. As a consequence, it is rescaled later without consequence for the proof (see remark 3.18 in the proof of proposition 3.17).*

Using Stokes theorem and Gronwall lemma, as in section 2.2.2, the energy of u_{n+1} satisfies:

$$E_{u_{n+1}}(\mathcal{S}_\tau) \leq \tilde{C}\exp(\tilde{C}\epsilon)\left(||\theta||^2_{H^1(\mathcal{I}^+)} + \left(\sup_{\tau\in[0,\epsilon]} E_{u_n}(\mathcal{S}_\tau)\right)^3\right).$$

For $n = 0$, we have:

$$E_{u_0}(\mathcal{S}_\tau) \leq \tilde{C}\exp(\tilde{C}\epsilon)||\theta||^2_{H^1(\mathcal{I}^+)}.$$

We denote by (C_n) the sequence defined by:

$$C_n = \sup_{\tau\in[0,\epsilon]}\left\{E_{u_n}(\mathcal{S}_\tau)\right\}.$$

This sequence satisfies the inequality:

$$\forall n\in\mathbb{N}, C_{n+1} \leq \underbrace{\tilde{C}\exp(\tilde{C}\epsilon)}_{\alpha}\left(C_n^3 + \underbrace{||\theta||^2_{H^1(\mathcal{I}^+)}}_{\beta}\right) \text{ with } C_0 \leq \tilde{C}\exp(\tilde{C}\epsilon)||\theta||^2_{H^1(\mathcal{I}^+)}.$$

Let us then consider the sequence $(c_n)_n$ defined by:

$$\begin{cases} c_0 &= \alpha\beta \\ c_{n+1} &= \alpha\left(c_n^3 + \beta\right). \end{cases}$$

The purpose is to choose correctly $||\theta||_{H^1(\mathcal{I}^+)}$ such that the sequence is bounded. The function $x \mapsto \alpha\left(x^3+\beta\right)$ has three fixed points provided that the discriminant of the polynomial $X^3 - \frac{1}{\alpha}X + \beta$ satisfies:

$$\beta^2 - \frac{4}{27\alpha^3} < 0, \text{ ie } \frac{4}{27} > (\tilde{C}\exp(\tilde{C}\epsilon))^3||\theta||^4_{H^1(\mathcal{I}^+)}. \tag{3.6}$$

Using Cardano's formulae, its three zeros are:

$$\lambda_0 = \sqrt{\frac{4}{3\alpha}}\cos\left(\frac{1}{3}\arccos\left(-\sqrt{\frac{27\beta^2\alpha^3}{4}}\right)\right) \tag{3.7}$$

$$\lambda_1 = \sqrt{\frac{4}{3\alpha}}\cos\left(\frac{1}{3}\arccos\left(-\sqrt{\frac{27\beta^2\alpha^3}{4}}\right)+\frac{2\pi}{3}\right) \tag{3.8}$$

$$\lambda_2 = \sqrt{\frac{4}{3\alpha}}\cos\left(\frac{1}{3}\arccos\left(-\sqrt{\frac{27\beta^2\alpha^3}{4}}\right)+\frac{4\pi}{3}\right). \tag{3.9}$$

Since

$$\begin{array}{rcl} \frac{1}{3}\arccos\left(-\sqrt{\frac{27\beta^2\alpha^3}{4}}\right) & \in & \left[\frac{\pi}{6},\frac{\pi}{3}\right] \\ \frac{1}{3}\arccos\left(-\sqrt{\frac{27\beta^2\alpha^3}{4}}\right)+\frac{2\pi}{3} & \in & \left[\frac{5\pi}{6},\pi\right] \\ \frac{1}{3}\arccos\left(-\sqrt{\frac{27\beta^2\alpha^3}{4}}\right)+\frac{4\pi}{3} & \in & \left[\frac{3\pi}{2},\frac{5\pi}{3}\right], \end{array}$$

these roots can be compared as follows:

$$\lambda_1 < 0 < \lambda_2 < \lambda_0.$$

The fixed points λ_0 and λ_1 are repulsive whereas the fixed point λ_2 is attractive. As a consequence, if the (positive) initial condition c_0 is below the positive repulsive fixed point (the greater fixed point λ_0) of the function $x \mapsto \alpha\left(x^3+\beta\right)$, that is to say if

$$\begin{array}{c} \sqrt{\frac{4}{3\alpha}}\cos\left(\frac{1}{3}\arccos\left(-\sqrt{\frac{27\beta^2\alpha^3}{4}}\right)\right) \geq \alpha\beta \\ 3\cos\left(\frac{1}{3}\arccos\left(-\sqrt{\frac{27\beta^2\alpha^3}{4}}\right)\right) \geq \sqrt{\frac{27\beta^2\alpha^3}{4}}, \end{array}$$

the sequence (c_n) converges towards λ_2. This inequality is always satisfied as soon as:

$$\beta^2 - \frac{4}{27\alpha^3} < 0.$$

As a consequence, assuming that

$$\frac{4}{27(\tilde{C}\exp(\tilde{C}\epsilon))^3} > ||\theta||^4_{H^1(\mathcal{I}^+)},$$

the sequence (c_n) converges to the remaining attractive fixed point λ_2; (c_n) is bounded and so is (C_n), which is the expected result.

Another useful consequence of the convergence of the sequence (c_n) is the following. The limit of this sequence satisfies:

$$\lambda_2 \leq \sqrt{\frac{4}{3\alpha}} = \sqrt{\frac{4}{3\tilde{C}\exp(\tilde{C}\epsilon)}}.$$

As a consequence, there exists a integer n_0 such that:

$$\forall n \geq n_0,\ \sup_{\tau\in[0,\epsilon]}\{E_{u_n}(\mathcal{S}_\tau)\} \leq c_n \leq 2\sqrt{\frac{4}{3\tilde{C}\exp(\tilde{C}\epsilon)}}.\blacklozenge \tag{3.10}$$

Proposition 3.17. *The sequence (u_n) converges on $\mathcal{R}$ in $C^0([0,\epsilon],H^1(\mathcal{S}_\tau))$, that is to say in the norm $\left(\sup_{\tau\in[0,\epsilon]}||u||^2_{H^1(\mathcal{S}_\tau)}\right)$.*

Proof. The method is exactly the same as in the previous proposition. Let n be a positive integer and consider the energy tensor associated with the linear wave equation for the function δ_n

$$T_{ab} = \hat{\nabla}_a \delta_n \hat{\nabla}_b \delta_n + \hat{g}_{ab}\left(-\frac{1}{2}\nabla_c \delta_n \nabla^c \delta_n + \frac{\delta_n^2}{2}\right).$$

The energy associated with a time slice $\mathcal{S}_\tau$ is written as in the previous proposition:

$$E_{\delta_n}(\mathcal{S}_\tau) = \int_{\mathcal{S}_\tau}\left(\frac{1}{2}\sum_{i=0,1,2,3}(e_{\mathbf{i}}^a \hat{\nabla}\delta_n)^2 + \frac{\delta_n^2}{2}\right)\mathrm{d}\mu_{\mathcal{S}_\tau}$$

and it is equivalent to $\int_{\mathcal{S}_\tau} i^\star_{\mathcal{S}_\tau}(\star e_0^a T_{ab})$ (with constant which only depends on the geometric data, b and the Killing form of e_0^a).

Finally, the error term is:

$$\hat{\nabla}^a\left(e_0^b T_{ab}\right) = \hat{\nabla}^{(a} e_0^{b)} T_{ab} + \delta_n e_0^a \nabla_a \delta_n - \frac{1}{6}\mathrm{Scal}_{\hat{g}} \delta_n e_0^a \nabla_a \delta_n - b(e_0^a \nabla_a \delta_n)\delta_{n-1}(u_n^2 + u_n u_{n-1} + u_{n-1}^2).$$

and can be estimated in absolute value by:

$$\int_{\mathcal{S}_\tau} |\hat{\nabla}^a e_0^b T_{ab}|\mu_{\mathcal{S}_\tau} \leq C E_{\delta_n}(\mathcal{S}_\tau) + 2\int_{\mathcal{S}_\tau} \delta_{n-1}^2(u_n^4 + u_{n-1}^4)\mu_{\mathcal{S}_\tau}.$$

where C is a positive constant depending on $\sup(|\mathrm{Scal}_{\hat{g}}|)$, $\sup\left(||\hat{\nabla}^{(a} e_0^{b)}||\right)$, $\sup(|b|)$ and the foliation $\mathcal{S}_\tau$.

Using Hölder inequality and proposition 3.4 for the foliation $\mathcal{S}_\tau$, the non-linearity in the error term is estimated by:

$$\begin{aligned}\int_{\mathcal{S}_\tau} \delta_{n-1}^2 u_n^4 \mu_{\mathcal{S}_\tau} &\leq \left(\int_{\mathcal{S}_\tau} \delta_{n-1}^6 \mu_{\mathcal{S}_\tau}\right)^{\frac{1}{3}}\left(\int_{\mathcal{S}_\tau} u_n^6 \mu_{\mathcal{S}_\tau}\right)^{\frac{2}{3}} \\ &\leq K_{Sob^6}||\delta_n||^2_{H^1(\mathcal{S}_\tau)}||u_n||^4_{H^1(\mathcal{S}_\tau)}.\end{aligned}$$

The same inequality holds for $\delta_{n-1}^2 u_{n-1}^4$.

Finally, there exists a constant K, such that:

$$\begin{aligned}&\int_{\mathcal{S}_\tau} |\hat{\nabla}^a e_0^b T_{ab}|\mu_{\mathcal{S}_\tau} \\ &\leq K\left(\int_0^t E_{\delta_n}(\mathcal{S}_r)\mathrm{d}r + \epsilon\left(\sup_{r\in[0,\epsilon]}\left(E_{\delta_{n-1}}(\mathcal{S}_r)\right)\right)\sup_{k\geq n-1}\left(\sup_{r\in[0,\epsilon]}\left(E_{u_n}(\mathcal{S}_r)\right)\right)^4\right).\end{aligned}$$

Stokes theorem is then applied beween $\mathcal{S}_\tau$ and $\mathscr{I}^+$: since the characteristic data for δ_n are zero, the only remaining term is the energy on the surface $\mathcal{S}_\tau$. Modulo a constant which only depends on the same data as the constant $\tilde{C}$, the integral inequality holds:

$$E_{\delta_n}(\mathcal{S}_\tau) \leq \tilde{K}\left(\int_0^\tau E_{\delta_n}(\mathcal{S}_r)\mathrm{d}r + \epsilon\left(\sup_{r\in[0,\epsilon]}\left(E_{\delta_{n-1}}(\mathcal{S}_r)\right)\right)\sup_{k\geq n-1}\left(\sup_{r\in[0,\epsilon]}\left(E_{u_n}(\mathcal{S}_r)\right)\right)^4\right),$$

for some contant $\tilde{K}$ and, using Gronwall's lemma, we get:

$$E_{\delta_n}(\mathcal{S}_\tau) \leq \tilde{K}\exp(\tilde{K}\epsilon)\epsilon\left(\sup_{r\in[0,\epsilon]}\left(E_{\delta_{n-1}}(\mathcal{S}_r)\right)\right)\sup_{k\geq n-1}\left(\sup_{r\in[0,\epsilon]}\left(E_{u_n}(\mathcal{S}_r)\right)\right)^4.$$

Remark 3.18. *The constant $\tilde{K}$, as the constant $\tilde{C}$ depends only the foliation $\mathcal{S}_\tau$, its scalar curvature, the Killing form of e_0^a and the supremum of b. As a consequence, up to a rescaling of $\tilde{C}$ or $\tilde{K}$, these constants can be chosen to be equal.*

Finally, the sequence (δ_n) satisfies:

$$\sup_{r\in[0,\epsilon]} (E_{\delta_n}(\mathcal{S}_\tau)) \leq \tilde{C}\exp(\tilde{C}\epsilon)\epsilon \left(\sup_{r\in[0,\epsilon]} \left(E_{\delta_{n-1}}(\mathcal{S}_r)\right)\right) \sup_{k\geq n-1} \left(\sup_{r\in[0,\epsilon]} (E_{u_n}(\mathcal{S}_r))\right)^4 .$$

Using inequality (3.10), we have:

$$\begin{aligned} \tilde{C}\exp(\tilde{C}\epsilon)\epsilon \sup_{k\geq n-1} \left(\sup_{r\in[0,\epsilon]} (E_{u_n}(\mathcal{S}_r))\right)^4 &\leq \tilde{C}\epsilon\exp(\tilde{C}\epsilon)\left(2\sqrt{\frac{4}{3\tilde{C}\exp(\tilde{C}\epsilon)}}\right)^2 \\ &\leq \frac{16}{3}\epsilon. \end{aligned} \tag{3.11}$$

Since ϵ is smaller than $\frac{1}{16}$, the sequence (δ_n) is then eventually contracting. The series of $(\sup_{r\in[0,\epsilon]}\left(||\delta_n||_{H^1(\mathcal{S}_r)}\right)^2)_n$ converges in the norm $\left(\sup_{\tau\in[0,\epsilon]}||u||^2_{H^1(\mathcal{S}_\tau)}\right)$, that is to say in $C^0([0,\epsilon],H^1(\mathcal{S}_\tau))$, and so does the sequence (u_n) .

End of the proof of theorem 3.12. The proof of the local existence is a direct consequence of the fact the sequence $(u_n)_{n\in\mathbb{N}}$ converges strongly on $\mathcal{R}$ for the norm

$$\sup_{r\in[0,\epsilon]}\left(||u||_{H^1(\mathcal{S}_r)}\right).$$

Let u be the limit of the sequence $(u_n)_n$. The only remaining thing to show is that the limit solves the problem of theorem 3.12. It is clear that u satisfies the initial conditions since all the functions u_n are identically equal to θ on $\mathcal{I}^+$. Finally, when noticing that u is in $H^1(\mathcal{R})$ which is continuously embedded in $L^3(\mathcal{R})$ (since $\mathcal{R}$ is four dimensional), the sequence $(u_n^3)_n$ converges in $L^1(\mathcal{R})$ and, as a consequence, in the distribution sense. u then satisfied the equation

$$\hat{\Box}u + \frac{1}{6}\mathrm{Scal}_{\hat{g}}u + bu^3 = 0$$

in the distribution sense.⧫

3.3 Global characteristic Cauchy problem

A global Cauchy problem is finally derived in two steps:

1. a preliminary result about the Cauchy problem for a hypersurface in the future of Σ_0 whose past contains i^0;

2. the characteristic Cauchy problem is then solved for small data with compact support which contains neither i^0 nor i^+ and then extended to functions in $H^1(\mathcal{I}^+)$.

3.3.1 Global Cauchy problem for compactly supported data

Starting from the same data θ in $H^1(\mathcal{I}^+)$ whose support contains neither i^+ nor i^0, the solution obtained in theorem 3.1 is extended to the future of Σ_0 by using density results and continuity of the propagator. The purpose of this section is to show that is it possible, starting from the hypersurface $\mathcal{S}$ in the future of Σ_0, to obtain a solution down to Σ_0 despite the singularity in i^0.

Proposition 3.19. *Let V^a be an orthogonal and normalized vector field to the uniformly spacelike hypersurface $\mathcal{S}$.*
The non-linear problem on $\mathcal{S}$:

$$\begin{cases} \hat{\Box} v + \frac{1}{6} Scal_{\hat{g}} v + bv^3 = 0 \\ v\big|_{\Sigma_0} = \xi \in H_0^1(\mathcal{S}) \\ V^a \hat{\nabla}_a v\big|_{\Sigma_0} = \zeta \in L^2(\mathcal{S}) \end{cases}$$

admits a global unique solution down to Σ_0 in $C^0(\mathbb{R}, H_0^1(\mathcal{S}))$.

Proof. The method consists in approximating the solution by solutions of the same problem with truncated data since the existence result for the Cauchy problem given by theorem 1.17 cannot be applied here directly because of the singularity in i^0. The uniqueness directly comes from theorem 3.6 and its corollary.

Let $(\chi_n)_{n\in\mathbb{N}}$ be a sequence of smooth functions with compact support in the interior of $\mathcal{S}$ such that:

$$\forall n \in \mathbb{N}, \mathrm{supp}\,(\chi_n) \subset \mathrm{supp}\,(\chi_{n+1}) \text{ and } \bigcup_{n\in\mathbb{N}} \mathrm{supp}\,(\chi_n) = \mathcal{S} \backslash \partial\mathcal{S}.$$

Let $(v_n)_{n\in\mathbb{N}}$ be the sequence defined by:

$$\begin{cases} \hat{\Box} v_n + \frac{1}{6} \mathrm{Scal}_{\hat{g}} v_n + bv_n^3 = 0 \\ v_n\big|_{\mathcal{S}} = \chi_n \xi \in H^1(\mathcal{S}). \\ V^a \hat{\nabla}_a v_n\big|_{\mathcal{S}} = \chi_n \zeta \in L^2(\mathcal{S}). \end{cases}$$

Since the data are with compact support in the interior of $\mathcal{S}$, their pasts do not intersect $\mathcal{I}^+$ and, as a consequence, i^0.

Using proposition 3.9, this sequence converges towards a function v in the past of $\mathcal{S}$ down to Σ_0 for the $L^\infty H^1$ norm. This function clearly satisfies the initial conditions:

$$v\big|_{\mathcal{S}} = \xi \text{ on } \mathcal{S} \text{ and } V^a \hat{\nabla}_a v\big|_{\mathcal{S}} = \zeta.$$

Furthermore, proposition 3.9 also gives convergence in $H^1(J^-(\mathcal{S}))$ and, using Sobolev embeddings, as a consequence, in $L^6(J^-(\mathcal{S}))$. v then satisfies the nonlinear wave equation in the distribution sense. ♦

Remark 3.20. *A direct consequence of this construction is that the trace of the solution of the Cauchy problem on Σ_0 is in $H_0^1(\Sigma_0)$*

3.3.2 Global characteristic Cauchy problem for small initial data in $H^1(\mathcal{I}^+)$

A global solution to the Goursat problem with compact which contains neither i^+ nor i^0 is then obtained by gluing solution of the local characteristic Cauchy problem and the solution of a well-chosen Cauchy problem on $\mathcal{S}$:

Proposition 3.21. *Let u be a solution to the Goursat problem for data θ with compact support which contains neither i^+ nor i^0.*
Then u can be extended from $\mathcal{S}$ down to Σ_0 in $C^0(\mathbb{R}, H^1(\mathcal{S}))$.

Proof. Consider the Cauchy problem on $\mathcal{S}$:

$$\left\{\begin{array}{l} \hat{\Box}v + \frac{1}{6}\mathrm{Scal}_{\hat{g}}v + bv^3 = 0 \\ v\big|_{\mathcal{S}} = u \in H^1(\mathcal{S}). \\ V^a\hat{\nabla}_a v\big|_{\mathcal{S}} = V^a\hat{\nabla}_a u \in L^2(\mathcal{S}). \end{array}\right.$$

According to proposition 3.19, this problem admits a global solution v down to Σ_0 in $C^0(\mathbb{R}, H^1(\mathcal{S}))$.
Finally, the function w defined piecewise by:

$$w = u \text{ on } J^+(S) \text{ and } w = v \text{ on } J^+(\Sigma_0) \cap J^-(S).$$

satisfies the Goursat problem:

$$\left\{\begin{array}{l} \hat{\Box}u + \frac{1}{6}\mathrm{Scal}_{\hat{g}}u + bu^3 = 0 \\ \phi\big|_{\mathcal{I}^+} = \theta \in H^1(\mathcal{I}^+) \end{array}\right. \quad \blacklozenge$$

Using proposition 3.21 and the continuity result, we can state the theorem of existence of the Goursat problem for small initial data:

Theorem 3.22. *Let us consider the nonlinear characteristic Cauchy problem on $\mathcal{I}^+$:*

$$\left\{\begin{array}{lcl} \hat{\Box}u + \frac{1}{6}Scal_{\hat{g}}u + bu^3 & = & 0 \\ u\big|_{\mathcal{I}^+} = \theta \in H^1(\mathcal{I}^+). & & \end{array}\right.$$

Then, for $||\theta||_{H^1(\mathcal{I}^+)}$ small enough, this problem admits a global unique solution down to the future of Σ_0 in $C^0(\mathbb{R}, H^1(\Sigma_0))$.

Remark 3.23. *As previously said in section 1.2.2 (see proposition 1.13), the singularity in i^+ is removable for Sobolev space in the sense that it is a regular point of a bigger manifold.*

Proof. The proof relies on the density of data with compact support in $H^1(\mathcal{I}^+)$ which does not neither i^0 nor i^+ in $H^1(\mathcal{I}^+)$ (proposition 1.13) and proposition 3.21.$\blacklozenge$

Remark 3.24. *As noticed above, the trace of the solution of the Goursat problem on Σ_0 is in $H^1_0(\Sigma)$.*

4 Construction of the scattering operator

The construction of the scattering operator can now be done by the mean of Cauchy problem on $\mathcal{I}^+$, $\mathcal{I}^-$ and Σ_0 via the composition of trace operators.

4.1 Existence and continuity of trace operators

The purpose of this section is to define trace operators for the solution of the wave equation on the hypersurfaces Σ_0 and $\mathcal{I}^+$. A symmetric construction can of course be realized on the past null infinity $\mathcal{I}^-$.

These trace operators are obtained using the following theorem ([82], p. 287):

Theorem 4.1. *Let M be a smooth compact manifold with piecewise C^1 boundary and consider the application T defined by:*

$$T : \left\{ \begin{array}{ccc} C^0(M) & \longrightarrow & C^0(\partial M) \\ f & \longmapsto & f\big|_{\partial M}. \end{array} \right.$$

Then, for all $s > \frac{1}{2}$, the operator T extends uniquely to a continuous map from $H^s(M)$ into $H^{s-\frac{1}{2}}(\partial M)$.

Existence theorems 1.17 and 3.22 give solutions to the initial (characteristic) problem in $H^1(\mathcal{J}^+(\Sigma_0))$. As a consequence, their traces on Σ_0 and $\mathcal{I}^+$ are respectively in $H^{\frac{1}{2}}(\Sigma_0)$ and $H^{\frac{1}{2}}(\mathcal{I}^+)$. Nonetheless, using the a priori estimates, they are in fact $H^1(\mathcal{I}^+)$ and $H^1(\Sigma_0)$.

Remark 4.2. *The singularity in i^+ is not a threat to the existence of a trace since the manifold and the metric can be extended with arbitrary regularity in a neighborhood of i^+. The problem with the singularity i^0 is avoided since the function spaces $H^1(\mathcal{I}^+)$ and $H^1_0(\Sigma_0)$ are the completions of smooth functions whose compact support does not contain i^0.*

Let us consider the trace operators:

$$T_0^+ := \left\{ \begin{array}{ccc} C_0^\infty(\Sigma_0) \times C_0^\infty(\Sigma_0) & \longrightarrow & H^1(\mathcal{I}^+) \\ (\theta, \tilde{\theta}) & \longmapsto & \phi\big|_{\mathcal{I}^+} \end{array} \right. \tag{4.1}$$

where ϕ is the unique solution of the problem:

$$\left\{ \begin{array}{l} \hat{\Box}\phi + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\phi + b\phi^3 = 0 \\ \phi\big|_{\Sigma_0} = \theta \in C_0^\infty(\Sigma_0) \\ \hat{T}^a\hat{\nabla}_a\phi\big|_{\Sigma_0} = \tilde{\theta} \in C_0^\infty(\Sigma_0) \end{array} \right.$$

obtained by theorem 1.17 and

$$T_+^0 := \left\{ \begin{array}{ccc} \mathcal{E} & \longrightarrow & H_0^1(\Sigma_0) \times L^2(\Sigma_0) \\ \theta & \longmapsto & (\phi\big|_{\Sigma_0}, (\hat{T}^a\hat{\nabla}_a\phi)\big|_{\Sigma_0}) \end{array} \right. \tag{4.2}$$

where $\mathcal{E}$ is the set of smooth functions with compact support which contains neither i^+ nor i^0 and ϕ is the unique solution of the problem:

$$\left\{ \begin{array}{lcl} \hat{\Box}\phi + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\phi + b\phi^3 & = & 0 \\ \phi\big|_{\mathcal{I}^+} = \theta \in C_0^\infty(\mathcal{I}^+) & & \end{array} \right.$$

obtained by theorem 3.22.

Remark 4.3. *The operator T_+^0 is not globally defined on $\mathcal{E}$ since theorem 3.22 only gives existence for small data. We denote by $\mathcal{B}^\infty_{\mathcal{I}^+}$ the trace of the open ball $\mathcal{B}_{\mathcal{I}^+}$ centered in zero in $H^1(\mathcal{I}^+) \cap \mathcal{E}$ on which T_+^0 is defined.*

These operators can be extended to $H^1(\Sigma_0)$ and $H^1(\mathcal{I}^+)$:

Proposition 4.4. *The operator T_0^+ can be extended to a locally Lipschitz operator from $H_0^1(\Sigma_0) \times L^2(\Sigma_0)$ to $H^1(\mathcal{I}^+)$.*
The operator T_+^0 can be extended to an Lipschitz operator from $\mathcal{B}_{\mathcal{I}^+} \subset H^1(\mathcal{I}^+)$ to $H_0^1(\Sigma_0) \times L^2(\Sigma_0)$.

Proof. The proof is done for the operator T_0^+ (it is exactly the same on the other side).

Let R be a positive constant. We denote by B_R the ball centered in 0 with radius R in $H^1(\Sigma_0) \times L^2(\Sigma_0)$ for the norm:

$$||\star||^2_{H^1(\Sigma_0)} + ||\star||^2_{H^1(\Sigma_0)}$$

Using theorem 3.6, this operator satisfies:

$$\forall (\theta, \tilde{\theta}, \xi, \tilde{\xi}) \in (C_0^\infty(\Sigma_0) \cap B_R))^4,$$
$$||T_0^+(\theta, \tilde{\theta}) - T_0^+(\xi, \tilde{\xi})||_{H^1(\mathcal{I}^+)} \leq C(R) \left(||\theta - \xi||^2_{H^1(\Sigma_0)} + ||\tilde{\theta} - \tilde{\xi}||^2_{L^2(\Sigma_0)} \right)^{\frac{1}{2}}.$$

As a consequence, since the smooth functions wit compact support in Σ_0 are dense in $H_0^1(\Sigma_0)$, it admits a unique locally-Lipschitz extension from $H_0^1(\Sigma_0) \times L^2(\Sigma_0)$ into $H^1(\mathcal{I}^+)$.

The same proof holds for the operator T_+^0. The only difference is the that, due to theorem 3.22 which only provides us with solutions for small data, this operator is defined on an open all of $H^1(\mathcal{I}^+)$ and, as a consequence, is globally Lipschitz. ♦

As already noted at the beginning of this section, a similar construction can be achieved on the past null infinity: there exist two trace operators T_-^0 and T_0^-, respectively locally Lipschitz and Lipschitz defined by:

$$T_0^- := \left\{ \begin{array}{ccc} H_0^1(\Sigma_0) \times L^2(\Sigma_0) & \longrightarrow & H^1(\mathcal{I}^-) \\ (\theta, \tilde{\theta}) & \longmapsto & \phi\big|_{\mathcal{I}^+} \end{array} \right. \tag{4.3}$$

where ϕ is the unique solution of the problem:

$$\left\{ \begin{array}{l} \hat{\Box}\phi + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\phi + b\phi^3 = 0 \\ \phi\big|_{\Sigma_0} = \theta \in H_0^1(\Sigma_0) \\ \hat{T}^a \hat{\nabla}_a \phi\big|_{\Sigma_0} = \tilde{\theta} \in L^2(\Sigma_0) \end{array} \right.$$

obtained by theorem 1.17 and

$$T_-^0 := \left\{ \begin{array}{ccc} \mathcal{B}_{\mathcal{I}^-} & \longrightarrow & H_0^1(\Sigma_0) \times L^2(\Sigma_0) \\ \theta & \longmapsto & (\phi\big|_{\Sigma_0}, (\hat{T}^a \hat{\nabla}_a \phi)\big|_{\Sigma_0}) \end{array} \right. \tag{4.4}$$

where $\mathcal{B}_{\mathcal{I}^-}$ is an open ball in $H^1(\mathcal{I}^-)$ and ϕ is the unique solution of the problem:

$$\left\{ \begin{array}{l} \hat{\Box}\phi + \frac{1}{6}\mathrm{Scal}_{\hat{g}}\phi + b\phi^3 \quad = \quad 0 \\ \phi\big|_{\mathcal{I}^-} = \theta \in H^1(\mathcal{I}^-) \end{array} \right.$$

obtained by theorem 3.22.

4.2 Conformal scattering operator

Finally, the conformal scattering operator is obtained as the composition of two trace operators. Following the idea of of Friedlander in [40] and applied by Mason-Nicolas in [62] for the Dirac and wave equations, the conformal scattering operator S is defined by the composition of the operators T_-^0 and T_0^+:

$$S = T_0^+ \circ T_-^0 : H^1(\mathcal{I}^-) \longrightarrow H^1(\mathcal{I}^-) \tag{4.5}$$

and its inverse is given by

$$S^{-1} = T_0^- \circ T_+^0 : H^1(\mathcal{I}^+) \longrightarrow H^1(\mathcal{I}^-) \tag{4.6}$$

These operators are not defined globally on $H^1(\mathcal{I}^+)$ or $H^1(\mathcal{I}^+)$ due to the restrictions imposed by theorem 3.22. Its domain of definition in our context is obtained from the domains of definition of T_-^0 and T_+^0 as follows: let $\mathcal{B}$ be the open set defined by:

$$\mathcal{B} = T_-^0(\mathcal{B}_{\mathcal{I}^-}) \cap T_+^0(\mathcal{B}_{\mathcal{I}^+}).$$

The images of $\mathcal{B}$ under T_0^- and T_0^+ give the domains of definition of S and S^{-1}, respectively. Finally, the following existence result for the conformal scattering operator can be stated:

Theorem 4.5 (Scattering operator)**.** *The operator S is an inversible, Lipschitz operator from $T_0^-(\mathcal{B})$ in $H^1(\mathcal{I}^-)$ into $T_0^+(\mathcal{B})$ in $H^(\mathcal{I}^+)$. This operator is called conformal scattering operator.*

Proof. The proof is an immediate consequence of proposition 4.4.♦

Remark 4.6. *The conformal scattering operator was introduced to avoid the use of the spectral theory which requires the metric to be static. It is nonetheless possible to talk about geometric scattering at least in the Schwarzschild part of the manifold and wonder wether it is possible to establish an equivalence in this region. Some answers to this question can be found in [62] (section 4.2) for the Dirac and Maxwell equations.*

5 Alternative approach for the a priori estimates

The use of a timelike vector field for the unphysical metric to obtain scattering seems a little unnatural though it is the relevant technical choice to obtain such estimates. An attempt is made to obtain the same estimates working instead with a timelike vector field for the physical metric g. The estimates which would be obtained are then weaker than the ones obtained using Hörmander's method (see below remark 5.3).

Though this attempt works quite well in the Schwarzschild part, we did not succeed to obtain the same type of estimates in the neighborhood of $\mathcal{I}$. The main problem is that we do not control the asymptotic behavior of the Killing form of a given vector field. There exists nonetheless a vector field which was studied near $\mathcal{I}^+$ in the context of the Einstein equations and peeling and successfully used for the linear wave equation: the gradient (with respect to the metric $\hat{g}$) of the conformal factor Ω. Its asymptotic behavior is described in [79] (chapter 6.8 and 9) before Penrose introduced peeling. As mentioned above, Mason–Nicolas used it in [62] to obtain estimates for the Maxwell and Dirac equations near $\mathcal{I}^+$.

The framework of this section is exactly the same as the one of section 2.

5.1 A priori estimates near i^0

The vector field in this section is chosen to be the Killing vector field $\partial_t = \partial_u$. As in section 2.1, we still work with the same foliation by the same hypersurfaces $\mathcal{H}_s$. The method is exactly the same: using lemma 1.5, an energy equivalence is established; this equivalence holds provided that ϵ is chosen wisely. The second step consists in establishing an integral inequality and, using Gronwall estimates.

This vector field T^a has the following property:

- it is know to be Killing for the conformal metric:

$$\hat{\nabla}^{(a}T^{b)} = 0;$$

- its norm for the unphysical metric:

$$\hat{g}_{ab}T^aT^b = R^2(1-2mR).$$

This vanishes at $\mathscr{I}^+$; the vector T is then tangent and normal to $\mathscr{I}^+$.

The first step consists in calculating the restriction of the energy 3-form to $\mathcal{H}_s$:

Proposition 5.1. *The restriction of the energy 3-form T^aT_{ab} to*

- *the hypersurface $\mathcal{H}_s$ is:*

$$i^\star_{\mathcal{H}_s}(\star T^aT_{ab}) = \Bigg\{ (\partial_u\phi)^2 + R^2(1-2mR)\partial_u\phi\partial_R\phi$$
$$+\frac{r^*R^2(1-2mR)}{2|u|}\left(R^2(1-2mR)(\partial_R\phi)^2 + |\hat{\nabla}_{\mathbb{S}^2}\phi|^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{2}\right)\Bigg\} du\wedge d\omega_{\mathcal{S}^2}.$$

- *the hypersurface S_u is:*

$$i^\star_{S_u}(\star T^aT_{ab}) = \int_{S_u}(\partial_R\phi)^2\, dR\wedge\omega_{\mathbb{S}^2}.$$

Proof. The energy on the spacelike hypersurface $\mathcal{H}_s = \{u + sr^* = 0\}$ is:

$$\begin{aligned} E(\mathcal{H}_s) &= \int_{\mathcal{H}_s} T^aT_{ab}\star \mathrm{d}^3x^b \\ &= \int_{\mathcal{H}_s}\partial_u\phi\nabla_b u\star\mathrm{d}^3x^b + \int_{\mathcal{H}_s}\left(-\frac{1}{2}\hat{\nabla}_c\phi\hat{\nabla}^c\phi + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right))T_b\star\mathrm{d}^3x^b \\ &= \int_{\mathcal{H}_s}\left(-\frac{1}{2}\hat{\nabla}_c\phi\hat{\nabla}^c\phi + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\mathrm{d}R\wedge\mathrm{d}\omega_{\mathbb{S}^2} \\ &-\int_{\mathcal{H}_s}\partial_u\phi\partial_R\phi(\partial_u\lrcorner\mathrm{d}\mu_{vol}) - \int_{\mathcal{H}_s}\partial_u\phi(-\partial_u\phi + R^2(1-2mR)\partial_R\phi)(\partial_R\lrcorner\mathrm{d}\mu_{vol}) \\ &-\int_{\mathcal{H}_s}\partial_u\phi(\nabla_{\mathbb{S}^2}\phi\lrcorner\mathrm{d}\mu_{vol}) \end{aligned}$$

Since

- $\partial_u\lrcorner\mathrm{d}\mu_{vol} = \mathrm{d}R\wedge\mathrm{d}\omega_{\mathbb{S}^2}$ and $\partial_R\lrcorner\mathrm{d}\mu_{vol} = -\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}$
- $\mathrm{d}R|_{\mathcal{H}_s} = \frac{r^*R^2(1-2mR)}{|u|}\mathrm{d}u|_{\mathcal{H}_s}$
- $\nabla_{\mathbb{S}^2}\phi$ is tangential to $\mathcal{H}_s$, and so $\nabla_{\mathbb{S}^2}\phi\lrcorner\mathrm{d}\mu_{vol}$ is transverse to $\mathcal{H}_s$

it remains:

$$E(\mathcal{H}_s) = \int_{\mathcal{H}_s} \frac{r^* R^2(1-2mR)}{|u|}\left(-\frac{1}{2}\hat{\nabla}_c\phi\hat{\nabla}^c\phi + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\mathrm{d}u \wedge \mathrm{d}\omega_{\mathbb{S}^2}$$

$$-\int_{\mathcal{H}_s} \frac{r^* R^2(1-2mR)}{|u|}\partial_u\phi\partial_R\phi\mathrm{d}u \wedge \mathrm{d}\omega_{\mathbb{S}^2} + \int_{\mathcal{H}_s}\partial_u\phi(\partial_u\phi + R^2(1-2mR)\partial_R\phi)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathbb{S}^2}$$

$$= \int_{\mathcal{H}_s}(\partial_u\phi)^2 + R^2(1-2mR)\partial_u\phi\partial_R\phi - \frac{r^* R^2(1-2mR)}{|u|}\partial_u\phi\partial_R\phi$$

$$\frac{r^* R^2(1-2mR)}{|u|}\left(-\frac{1}{2}\left(-R^2(1-2mR)\partial_R\phi^2 - 2\partial_R\phi\partial_u\phi - |\hat{\nabla}_{\mathbb{S}^2}\phi|^2\right) + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathcal{S}^2}$$

$$= \int_{\mathcal{H}_s}(\partial_u\phi)^2 + R^2(1-2mR)\partial_u\phi\partial_R\phi +$$

$$\frac{r^* R^2(1-2mR)}{2|u|}\left(R^2(1-2mR)(\partial_R\phi)^2 + |\hat{\nabla}_{\mathbb{S}^2}\phi|^2 + \frac{\phi^2}{2} + b\frac{\phi^4}{2}\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathcal{S}^2}.$$

The other inequality is straightforward when noticing that the vector field ∂_u satisfies:

$$\partial_u^a = -\hat{\nabla}^a R + R^2(1-2mR)\hat{\nabla}^a u.$$

and, as a consequence, since the restriction of du to S_u vanishes:

$$i^\star_{S_u}(\star\partial_u) = 0.$$

Finally, using the decomposition of $\nabla_b u$, we obtain:

$$i^\star_{S_u}(\star T^a T_{ab}) = \int_{S_u}(\partial_u\phi)^2\mathrm{d}R\wedge\mathrm{d}\omega_{\mathcal{S}^\in} \blacklozenge$$

As already mentioned, the next step consists in establishing the energy equivalence in a well-chosen neighborhood of i^0:

Proposition 5.2. *There exists u_0, $|u_0|$ large enough, such that the following equivalence holds:*

$$\int_{\mathcal{H}_s} i^\star_{\mathcal{H}_s}(\star T^a T_{ab}) \approx \int_{\mathcal{H}_s}(\partial_u\phi)^2 + \frac{R}{u}\left(R^2(\partial_R\phi)^2 + |\hat{\nabla}_{\mathcal{S}^2}\phi|^2 + \frac{\phi^2}{2} + b\phi^4\right)du\wedge d\omega_{\mathcal{S}^2}.$$

This energy is denoted by $E(\mathcal{H}_s)$.

Remark 5.3. *The only remaining term when $R = 0$ (or $s = 1$) is the L^2 norm of the derivatives tangential to $\mathcal{I}^+$:*

$$\int_{\mathcal{H}_s}(\partial_u\phi)^2 du\wedge d\omega_{\mathcal{S}^2}.$$

This energy is weaker than the one obtained when using Hörmander's method (see proposition 2.3).

Proof. The proof is based on exactly the same method as in section 2: using estimates on the asymptotics of the coordinates in the neighborhood of i^0 given by lemma 1.5, the following inequalities hold.

Since $1-2mR<1$ and $Rr^*<1+\epsilon$,

$$\int_{\mathcal{H}_s} i^\star_{\mathcal{H}_s}\left(\star T^a T_{ab}\right) \le \int_{\mathcal{H}_s} (\partial_u\phi)^2 + R^2\partial_u\phi\partial_R\phi$$
$$+(1+\epsilon)\frac{R}{|u|}\left(R^2(\partial_R\phi)^2+|\hat\nabla_{\mathbb{S}^2}\phi|^2+\frac{\phi^2}{2}+b\frac{\phi^4}{2}\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathcal{S}^2}$$
$$\le (1+\epsilon)\int_{\mathcal{H}_s}(\partial_u\phi)^2+R^2\partial_u\phi\partial_R\phi+\frac{R}{|u|}\left(R^2(\partial_R\phi)^2+|\hat\nabla_{\mathbb{S}^2}\phi|^2+\frac{\phi^2}{2}+b\frac{\phi^4}{2}\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathcal{S}^2}.$$

Furthermore, since $R|u|<1+\epsilon$

$$\int_{\mathcal{H}_s} R^2\partial_u\phi\partial_R\phi\mathrm{d}u\wedge\mathrm{d}\omega_{\mathcal{S}^2} \le \frac12\int_{\mathcal{H}_s}\left((\partial_u\phi)^2+R^4(\partial_R\phi)^2\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathcal{S}^2}$$
$$\le \frac12\int_{\mathcal{H}_s}\left((\partial_u\phi)^2+R|u|\frac{R}{|u|}R^2(\partial_R\phi)^2\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathcal{S}^2}$$
$$\le \int_{\mathcal{H}_s}\left((\partial_u\phi)^2+(1+\epsilon)\frac{R}{|u|}R^2(\partial_R\phi)^2\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathcal{S}^2}$$
$$\le (1+\epsilon)\int_{\mathcal{H}_s}\left((\partial_u\phi)^2+\frac{R}{|u|}R^2(\partial_R\phi)^2\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathcal{S}^2}$$

it remains:

$$\int_{\mathcal{H}_s} i^\star_{\mathcal{H}_s}\left(\star T^a T_{ab}\right)\le 2(1+\epsilon)^2E(\mathcal{H}_s)$$

On the other hand, since $Rr^*>1$ and $1-2mR>1-\epsilon$,

$$\int_{\mathcal{H}_s} i^\star_{\mathcal{H}_s}\left(\star T^a T_{ab}\right) \ge \int_{\mathcal{H}_s}(\partial_u\phi)^2+(1-\epsilon)R^2\partial_u\phi\partial_R\phi$$
$$+(1-\epsilon)\frac{R}{2|u|}\left((1-\epsilon)R^2(\partial_R\phi)^2+|\hat\nabla_{\mathbb{S}^2}\phi|^2+\frac{\phi^2}{2}+b\frac{\phi^4}{2}\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathcal{S}^2}$$

Since:

$$\int_{\mathcal{H}_s} R^2\partial_u\phi\partial_R\phi\mathrm{d}u\wedge\mathrm{d}\omega_{\mathcal{S}^2}\ge -\frac12\int_{\mathcal{H}_s}\left((\partial_u\phi)^2+R^4(\partial_R\phi)^2\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathcal{S}^2},$$

we get:

$$\int_{\mathcal{H}_s} i^\star_{\mathcal{H}_s}\left(\star T^a T_{ab}\right)\ge\int_{\mathcal{H}_s}\frac{1+\epsilon}{2}(\partial_u\phi)^2+$$
$$\frac{R}{4|u|}\left(\frac{1-3\epsilon+2\epsilon^2}{2}R^2(\partial_R\phi)^2+(1-\epsilon)\left(|\hat\nabla_{\mathbb{S}^2}\phi|^2+\frac{\phi^2}{2}+b\frac{\phi^4}{2}\right)\right)\mathrm{d}u\wedge\mathrm{d}\omega_{\mathcal{S}^2}$$
$$\ge C(\epsilon)E(\mathcal{H}_s)$$

where $C(\epsilon)=\min\left(\frac{1+\epsilon}{2},\frac{1-3\epsilon+2\epsilon^2}{8},\frac{1-\epsilon}{4}\right)$, which is positive for ϵ small enough.♦

u_0 being chosen, we can finally establish the energy equivalence in the neighborhood $\Omega^+_{u_0}$; we adopt the same notation $\Sigma_0^{u_0>}$ to designate the part of Σ_0 in $\Omega^+_{u_0}$:

Proposition 5.4. *The following energy equivalence holds in $\Omega^+_{u_0}$:*

$$E(\mathcal{H}_s)+E(S_{u_0})\approx E(\Sigma_0^{u_0>}).$$

Proof. We calculate the error term in the Stokes theorem:

$$\begin{aligned}\nabla^a(T^bT_{ab}) &= \nabla^{(a}T^{b)}T_{ab} - \frac{1}{6}\mathrm{Scal}_{\hat{g}}\phi\partial_u\phi + (\partial_u\phi)\phi + \partial_u b\frac{\phi^4}{4}\\ &= \nabla^{(a}T^{b)}T_{ab} + (1-2mR)\phi\partial_u\phi + \partial_u b\frac{\phi^4}{4}\end{aligned}$$

since ∂_u is Killing, the only remaining term is:

$$\nabla^a(T^bT_{ab}) = (1-2mR)\phi\partial_u\phi + \partial_u b\frac{\phi^4}{4}$$

We then apply Stokes theorem between the surface $\Sigma_0^{u_0>} = \mathcal{H}_1$, S_{u_0} and $\mathcal{H}_s$:

$$E(\mathcal{H}_s) - E(\mathcal{H}_1) + E(S_{u_0}) =$$
$$\int_1^s\int_{\Sigma_\tau} -\left((1-2mR)\phi\partial_u\phi + \partial_u b\frac{\phi^4}{4}\right)(r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{\frac{R}{|u|}}\mathrm{d}u\wedge\omega_{\mathbb{S}^2}\mathrm{d}\tau$$

The same change of parameter as in section 2.1 (see formulae (2.9) and (2.10)) is done. Using the decomposition induced by this choice, the error term can be estimated by:

$$\begin{aligned}&\left|\int_{\mathcal{H}\mathcal{T}(s)}\left((1-2mR)\phi\partial_u\phi + \partial_u b\frac{\phi^4}{4}\right)(r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{\frac{R}{|u|}}\mathrm{d}u\wedge\omega_{\mathbb{S}^2}\right|\\ &\leq (1+\epsilon)^{\frac{3}{2}}\int_{\Sigma_\tau}\sqrt{\frac{R}{|u|}}\left(|\phi\partial_u\phi| + \partial_u b|\frac{\phi^4}{4}\right)\mathrm{d}u\wedge\omega_{\mathbb{S}^2}\\ &\leq (1+\epsilon)^{\frac{3}{2}}\int_{\Sigma_\tau}\left(\left(\frac{R}{|u|}|\phi^2 + (\partial_u\phi)^2\right) + \sqrt{\frac{R}{|u|}}|\partial_u b|\frac{\phi^4}{4}\right)\mathrm{d}u\wedge\omega_{\mathbb{S}^2}\end{aligned}$$

We assume that, in the neighborhood $\Omega^+_{u_0}$:

$$|\partial_u b| \leq \sqrt{\frac{R}{|u|}}|b|$$

so that:

$$\begin{aligned}&\left|\int_{\Sigma_\tau}\left((1-2mR)\phi\partial_u\phi + \partial_u b\frac{\phi^4}{4}\right)(r^*R)^{\frac{3}{2}}(1-2mR)\sqrt{\frac{R}{|u|}}\mathrm{d}u\wedge\omega_{\mathbb{S}^2}\right|\\ &\leq (1+\epsilon)^{\frac{3}{2}}\int_{\Sigma_\tau}\left(\left(\frac{R}{|u|}|\phi^2 + (\partial_u\phi)^2\right) + \frac{R}{|u|}|\partial_u b|\frac{\phi^4}{4}\right)\mathrm{d}u\wedge\omega_{\mathbb{S}^2}\\ &\lesssim E(\mathcal{H}_s)\end{aligned}$$

The remainder of the proof is then exactly the same as the proof of proposition 2.8. ⧫

5.2 Away from i^0

We saw that the apriori estimates for the chosen vector fields in the neighborhood of i^0 can be achieved more easily than in the case of the Morawetz vector field, for the main reason that we choose to work with a Killing vector field, which avoids the problem of obtaining estimates for the Killing form of T^a. This problem arises here since no assumptions were made on the behavior of T^a in the neighborhood of $\mathcal{I}^+$.

There are two natural ways to come closer to $\mathcal{I}^+$. The first one consists in choosing the foliation associated with a timelike vector field for the metric g. The scattering result could then be interpreted as scattering in the usual sense (that is to say as a limit process in the physical time, when time goes to infinity). The main obstacle of this is that we cannot control the asymptotics of the Killing form associated with the gradient of the time function.

Another way is to use the foliation associated with the conformal factor Ω. This method is expected to work in the neighborhood of i^+. Nonetheless, we were not able to use it; a better understanding of the estimates obtained in [62] (lemma A.1, appendices A.2 and A.3) is certainly the proper tool to solve this question.

5.2.1 With a timelike vector field

Let t a smooth time function on the manifold M and $\hat{t}$ another time function on the compactified manifold $\hat{M}$. We denote:

$$T^a = \nabla^a t \text{ and } \hat{T}^a = \frac{\hat{\nabla}^a \hat{t}}{\hat{g}_{ab}\hat{\nabla}^a \hat{t}\hat{\nabla}^b \hat{t}}$$

their gradients for the metrics g and $\hat{g}$. Let finally (e_j^a) $(j = 0, \ldots, 3)$ be a global section of the fiber bundle of orthonormal frames with $e_0^a = \hat{T}^a$.

M admits a smooth foliation by the level hypersurfaces of the smooth function t; the hypersurface defined by $t =$ constant is denoted Σ_t and can be defined as the graph of a function $f_t = \hat{t}|_{\Sigma_t}$ from Σ_0, using the flow associated with $\hat{t}$, that is to say, if $\Phi_{\hat{t}}$ is the flow associated with T^a, the function f_t is implicitly defined as:

$$\forall p \in M, \hat{t}(p) = f_t(\Phi_{-\hat{t}(p)}(p)) \tag{5.1}$$

Using this description of Σ_t, the tangential derivatives along Σ_t are given by, for a smooth function ϕ on $\hat{M}$:

$$\forall x \in \Sigma_0, \forall j \in \{1,2,3\}, (\hat{\nabla}_j - \hat{\nabla}_j f_t \hat{\nabla}_0)\phi(f_t(x), x)$$

and the vector

$$N^a = e_0^a - \sum_{j\in\{1,2,3\}} \hat{\nabla}_j f_t e_j^a$$

defines a timelike and future oriented vector normal to the hypersurface Σ_t. The derivatives ∇_j must be understood as the derivatives on $T\Sigma_0$ along the vectors $\mathrm{d}\phi_{-\hat{t}(p)}(e_j)$ for the metric $\Omega^2\hat{g}|_{\Sigma_0}$. The tangential derivatives to the surface Σ_t are given by:

$$t_j^a = \hat{\nabla}_j f_t e_0^a - e_j^a \text{ for } j = 1 \ldots 3.$$

We finally introduce a vector field τ^a transverse to Σ_t:

$$\tau^a = e_0^a + \sum_{j\in\{1,2,3\}} \hat{\nabla}_j f_t e_j^a.$$

A straightforward calculation gives the following expression of the volume form in function of τ_a, N_a and t_j^a (for $j : 1,2,3$):

$$\mathrm{d}\mu[\hat{g}] = \frac{N_a \wedge t_a^1 \wedge t_a^2 \wedge t_a^3}{1 - \sum_{j=1,2,3}(\partial^j f_t)^2} \tag{5.2}$$
$$= \frac{\tau_a \wedge t_a^1 \wedge t_a^2 \wedge t_a^3}{1 + \sum_{j=1,2,3}(\partial^j f_t)^2}. \tag{5.3}$$

Remark 5.5. *It must be that the expression* (5.2) *is ill-suited for the following since the term* $1 - \sum_{j=1,2,3}(\partial^j f_t)^2$ *vanishes at null infinity. The calculation will then be performed using the transverse vector* τ *which is a null vector transverse to null infinity.*

We note:

$$\alpha = 1 + \sum_{j=1,2,3} (\partial^j f_t)^2 \text{ and } \beta = 1 - \sum_{j=1,2,3} (\partial^j f_t)^2.$$

It must be noticed that both N^a and $\nabla^a t$ are, future oriented and normal to the hypersurface, so there exists a positive function ξ such that:

$$\nabla^a t = T^a = \xi N^a$$

which can be expressed as:

$$\xi = \hat{g}_{ab}\nabla^a t e_0^b = \Omega^2 \mathrm{d}t(e_0).$$

The norm for the metric $\hat{g}$ is:

$$\hat{g}_{cd}T^c T^d = \xi^2 \beta.$$

The following lemma gives an explicit relation between ξ and the derivatives of f_t and $\hat{t}$:

Lemma 5.6. *The positive function* ξ *satisfies:*

$$\xi = \frac{\Omega^2}{\sqrt{\hat{g}_{ab}\partial_{\hat{t}}^a \partial_{\hat{t}}^b \frac{\partial f_t}{\partial t}}}$$

Proof. it relies on the relation (5.1) which is differentiated:

$$\mathrm{d}\hat{t} = \frac{\partial f_t}{\partial t}\mathrm{d}t + \mathrm{d}^x f_t \left(\mathrm{d}\Phi_{-\hat{t}(p)} - \frac{\partial \Phi_{-\hat{t}}}{\partial \hat{t}} \mathrm{d}\hat{t} \right)$$

The flow $\Phi_{\hat{t}}$ preserves by construction the orthogonality of e_0^a, so the differential preserves the orthogonality and the only remaining term, when applying the forms to e_0 is:

$$\mathrm{d}\hat{t}(e_0) = \frac{\partial f_t}{\partial t}\mathrm{d}t(e_0),$$

which gives the awaited result.♦

The volume form associated with the hypersurface is:

$$\mathrm{d}\mu_{\Sigma_t} = \bigwedge_{j\in\{1,2,3\}} \frac{\hat{\nabla}_j f_t e_a^0 - e_a^j}{((\hat{\nabla}_j f_t)^2 - 1)}$$

where e_a^j is the form obtained from the orthonormal basis e_j^a by lowering of the index a.

Proposition 5.7. *The energy on the slice Σ_t, $E(\Sigma_t)$, is given by:*

$$E(\Sigma_t) = \int_{\Sigma_t} \left(\left(T^a \hat{\nabla} \phi \right)^2 + ||T^a||^2_{\hat{g}} \left(-\frac{1}{2} \hat{g}_{cd} \hat{\nabla}^c \phi \hat{\nabla}^c \phi + \frac{\phi^2}{2} + b\frac{\phi^4}{4} \right) \right) \frac{t^1_a \wedge t^2_a \wedge t^3_a}{\xi^2 \alpha}$$

Remark 5.8. *1. The term $||T^a||^2$ is the main obstruction to obtain the a priori estimates, since it cannot be controlled. Assumptions must then be done to control it.*

2. This expression is practical when going to null infinity. Nonetheless, it could be interesting to express it as the energy on a slice Σ_t with the Riemanian metric induced by $\hat{g}$:

$$E(\Sigma_t) = \int_{\Sigma_t} \left(||\hat{\nabla}\phi||^2_{g|_{\Sigma_t}} + \frac{\phi^2}{2} + b\frac{\phi^4}{4} \right) \frac{t^1_a \wedge t^2_a \wedge t^3_a}{\xi^2 \alpha}$$

Proof. Let us consider the energy associated with T^a:

$$\begin{aligned} E(\Sigma_t) &= \int_{\Sigma_t} T^a T_{ab} \star \mathrm{d}^3 x^b \\ &= \int_{\Sigma_t} T^a \hat{\nabla}_a \phi \hat{\nabla}_b \phi \star \mathrm{d}^3 x^b + \left(-\frac{1}{2} \hat{\nabla}_c \phi \hat{\nabla}^c \phi + \frac{\phi^2}{2} + b\frac{\phi^4}{4} \right) T_b \star \mathrm{d}^3 x^b \qquad (5.4) \end{aligned}$$

Concerning the second part of the sum, the Hodge dual of $\nabla^a t$ is calculated with respect to the measure $\mathrm{d}\mu[\hat{g}]$. In order to have a non singular three-from when performing the calculation of the Hodge dual, the vector T^a is expressed in function of τ^a and the tangential derivatives to Σ_t. Noticing that

$$\tau^a + N^a = 2e^a_0 \text{ and } \tau^a + \sum_{j=1,2,3} \partial^j f_t t^a_j = \alpha e^a_0,$$

which gives:

$$N^a = \frac{\beta}{\alpha} \tau^a + \sum_{j=1}^{3} \frac{2}{\alpha} \partial^j f_t t^a_j$$

we have:

$$\begin{aligned} T^a &= \xi N^a \\ &= \frac{\xi\beta}{\alpha} \tau^a + \frac{2\xi}{\alpha} \sum_{j=1}^{3} \partial^j f_t t^a_j \end{aligned}$$

The second term of the expression of the vector is tangent to Σ_t whereas the first is transverse. So the Hodge dual of T^a, when restricted to $T\Sigma_t$, is:

$$\begin{aligned} T_a \star \mathrm{d}^3 x^a &= T^a \lrcorner \mathrm{d}\mu[\hat{g}] \\ &= \frac{\xi\beta}{\alpha} t^1_a \wedge t^2_a \wedge t^3_a. \end{aligned}$$

The calculation of $\star\hat{\nabla}_b \phi \mathrm{d}^3 x^b$ is done in the same way: splitting the vector $\hat{\nabla}^a \phi$ over the basis $(T^a, t^a_1, t^a_2, t^a_3)$:

$$\hat{\nabla}^a \phi = \frac{T^b \hat{\nabla}_b \phi}{\hat{g}_{cd} T^c T^d} T^a + \sum_{j=1,2,3} \frac{t^b_j \hat{\nabla}_b \phi}{\hat{g}_{cd} t^c_j t^d_j} t^a_j$$

its Hodge dual, when restricted to $T\Sigma_t$, is:

$$\begin{aligned}
\hat{\nabla}_a\phi \star \mathrm{d}^3x^a &= \left(\frac{T^b\hat{\nabla}_b\phi}{\hat{g}_{cd}T^cT^d}\right)\star T^a\mathrm{d}^3x^a \\
&= \left(\frac{T^b\hat{\nabla}_b\phi}{\hat{g}_{cd}T^cT^d}\right)\frac{\xi\beta}{\alpha^2}t^1_a\wedge t^2_a\wedge t^3_a \\
&= \left(\frac{T^b\hat{\nabla}_b\phi}{\xi\alpha^2}\right)t^1_a\wedge t^2_a\wedge t^3_a.
\end{aligned}$$

The energy (5.4) can finally be rewritten as:

$$\begin{aligned}
E(\Sigma_t) &= \int_{\Sigma_t}\left(\left(T^a\hat{\nabla}\phi\right)^2+\xi^2\beta\left(-\frac{1}{2}\hat{g}_{cd}\hat{\nabla}^c\phi\hat{\nabla}^d\phi+\frac{\phi^2}{2}+b\frac{\phi^4}{4}\right)\right)\frac{t^1_a\wedge t^2_a\wedge t^3_a}{\xi^2\alpha} \\
&= \int_{\Sigma_t}\left(\left(T^a\hat{\nabla}\phi\right)^2+||T^a||^2_{\hat{g}}\left(-\frac{1}{2}\hat{g}_{cd}\hat{\nabla}^c\phi\hat{\nabla}^d\phi+\frac{\phi^2}{2}+b\frac{\phi^4}{4}\right)\right)\frac{t^1_a\wedge t^2_a\wedge t^3_a}{\xi^2\alpha}\blacklozenge
\end{aligned}$$

5.2.2 With the conformal factor

We choose to work in this section with the conformal factor Ω. We assume in order to ensure the positivity of the energy that the vector $\hat{\nabla}^a\Omega$ is timelike on M. We denote

$$T^a = \hat{\nabla}^a\Omega.$$

the timelike future oriented vector field orthogonal in M to the hypersurfaces $\{\Omega = \textit{constant}\}$. The foliation induced by the conformal factor is denoted by Σ_Ω.

Remark 5.9. *It must be noted that the assumption that the gradient $\nabla\Omega$ is timelike is not necessarily satisfied. For instance, the usual choice for the Schwarzschild metric is $\frac{1}{r}$ whose gradient is spacelike behind the horizon of the black hole.*

All the calculations that were made in the previous section can be applied to that case; nonetheless, some calculations are made again considering this specific choice of time function.

The normal vector to the foliation is given by the gradient $\hat{\nabla}^a\Omega$:

$$\hat{\nabla}^a\Omega = \hat{\nabla}^0\Omega e^b_0 - \hat{\nabla}^1\Omega e^a_1 - \hat{\nabla}^2\Omega e^a_2 - \hat{\nabla}^3\Omega e^a_3,$$

where $\hat{\nabla}^j = e^b_j\hat{\nabla}_b$. The vector τ^a defined by:

$$\tau^a = \hat{\nabla}^0\Omega e^b_0 - \hat{\nabla}^1\Omega e^a_1 - \hat{\nabla}^2\Omega e^a_2 - \hat{\nabla}^3\Omega e^a_3,$$

is transverse to the foliation and will be used as an identifying vector field for the foliation. Finally, the tangent space to the foliation is spanned by:

$$t^a_j = \hat{\nabla}^j\Omega e^a_0 - \hat{\nabla}^0\Omega e^a_j.$$

The volume $\mu[\hat{g}]$ is split over these vectors as:

$$\begin{aligned}
\mu[\hat{g}] &= \frac{\hat{\nabla}_a\Omega\wedge t^1_a\wedge t^2_a\wedge t^3_a}{(\hat{\nabla}^0\Omega)^2\hat{g}_{cd}\nabla^c\hat{\Omega}\hat{\nabla}^d\Omega} \\
&= \frac{\tau_a\wedge t^1_a\wedge t^2_a\wedge t^3_a}{(\hat{\nabla}^0\Omega)^2\left((\hat{\nabla}^0\Omega)^2+(\hat{\nabla}^1\Omega)^2+(\hat{\nabla}^2\Omega)^2+(\hat{\nabla}^3\Omega)^2\right)}
\end{aligned}$$

We denote by α the quantity:

$$\alpha = (\hat{\nabla}^0\Omega)^2 + (\hat{\nabla}^1\Omega)^2 + (\hat{\nabla}^2\Omega)^2 + (\hat{\nabla}^3\Omega)^2.$$

For this choice of vector fields, the following proposition holds:

Proposition 5.10. *The energy on the slice Σ_Ω, $E(\Sigma_\Omega)$, is given by:*

$$\begin{aligned} E(\Sigma_\Omega) &= \int_{\Sigma_\Omega} \star\hat{\nabla}^a\Omega T_{ab} \\ &= \int_{\Sigma_\Omega} \left(\left(\hat{\nabla}^a\Omega\hat{\nabla}\phi\right)^2 + ||\hat{\nabla}^a\Omega||^2_{\hat{g}}\left(-\frac{1}{2}\hat{g}_{cd}\hat{\nabla}^c\phi\hat{\nabla}^d\phi + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\right)\frac{t^1_a\wedge t^2_a\wedge t^3_a}{(\hat{\nabla}^0\Omega)^2\alpha} \end{aligned}$$

Proof. The proof is the same as the proof of proposition 5.7. Nonetheless, since the choice of vector fields that we work with is slightly different, some of the calculations are made again. Noticing that:

$$\tau^a + \hat{\nabla}^a\Omega = 2e^a_0 \text{ and } \hat{\nabla}^0\Omega\tau^a + \sum_{j=1}^{3}\hat{\nabla}^j\Omega t^a_j = \alpha e^a_0$$

and, consequently,

$$\hat{\nabla}^a\Omega = \frac{||\hat{\nabla}\Omega||_{\hat{g}}}{\alpha}\tau^a + \sum_{i=1}^{3}\frac{2}{\alpha}\hat{\nabla}^j\Omega t^a_j,$$

we obtain the following expression of the energy:

$$\int_{\Sigma_\Omega}\star\nabla^a T_{ab} = \int_{\Sigma_\Omega}\left(\left(\hat{\nabla}^a\Omega\hat{\nabla}\phi\right)^2 + ||\hat{\nabla}^a\Omega||^2_{\hat{g}}\left(-\frac{1}{2}\hat{g}_{cd}\hat{\nabla}^c\phi\hat{\nabla}^d\phi + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\right)\frac{t^1_a\wedge t^2_a\wedge t^3_a}{(\hat{\nabla}^0\Omega)^2\alpha} \blacklozenge \tag{5.5}$$

Remark 5.11. *The quantities α and $\hat{\nabla}^0$ do not vanish at the considered compact region of the unphysical spacetime. The energy* (5.5) *is then uniformly equivalent, on this region, to:*

$$\int_{\Sigma_\Omega}\left(\left(\hat{\nabla}^a\Omega\hat{\nabla}\phi\right)^2 + ||\hat{\nabla}^a\Omega||^2_{\hat{g}}\left(-\frac{1}{2}\hat{g}_{cd}\hat{\nabla}^c\phi\hat{\nabla}^d\phi + \frac{\phi^2}{2} + b\frac{\phi^4}{4}\right)\right)t^1_a\wedge t^2_a\wedge t^3_a;$$

We now consider the error term:

$$\begin{aligned} \int_M \hat{\nabla}^a\left(\hat{\nabla}^b T_{ab}\right)\mu &= \int_M \left(1 + \frac{1}{6}\mathrm{scal}_{\hat{g}}\right)\left(\hat{\nabla}^a\Omega\hat{\nabla}_a\phi)\right) + \hat{\nabla}^a\Omega\hat{\nabla}_a b\frac{\phi^4}{4} \\ &\quad + \hat{\nabla}^{(a}\nabla^{b)}T_{ab} \end{aligned}$$

The main problem which must be dealt is that the Killing form of the gradient of Ω cannot be estimated.

Concluding remarks

There exist several possible extensions to this work:

- the case where the metric in the neighborhood of i^0 is the Kerr-Newman metric;
- the nonlinearity could be modified and the equation could, for instance, be quasilinear, or satisfy the null condition;
- following [63], these results could be extended to peeling results for the same cubic defocusing wave equation.

One of the main problems of general relativity is the construction of solutions to the Einstein equations. One intermediate step is to establish the same kind of result for the Yang-Mills equations.

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Résumé

L'étude présentée dans ce travail de thèse aborde deux aspects du problème de Cauchy caractéristique en relativité générale.
D'une part, une formule intégrale pour le problème de Cauchy caractéristique pour l'équation de Dirac est établie, généralisant les travaux de Penrose en espace-temps courbe. Ayant adapté le cadre fonctionnel pour obtenir une théorie des distributions adaptée à la structure algébriques des spineurs, le formalisme Geroch-Held-Penrose est utilisé pour décrire de la manière la plus précise possible la formule intégrale. La formule de Penrose en spin arbitraire sur l'espace-temps de Minkowski est retrouvée.
D'autre part, une théorie de scattering conforme pour une équation des ondes non linéaire conformément invariante sur un espace asymptotiquement simple est construite. En effectuant un rééchelonnement conforme, l'espace-temps est complété en lui ajoutant une frontière constituée de deux hypersurfaces caractéristiques représentant respectivement les extrémités passées et futures des géodésiques de type lumière. Le comportement asymptotique des champs s'obtient alors en considérant les traces des solutions de l'équation conforme sur ces bords. L'inversibilité des opérateurs de trace s'obtient alors en résolvant un problème de Cauchy caractéristique sur ce bord et l'opérateur de scattering conforme est obtenu par composition de ces opérateurs de trace.

Abstract

This work presents two aspects of the characteristic Cauchy problem in general relativity.
On the one hand, an integral formula for the characteristic Cauchy problem for the Dirac equation on a curved space-time is derived. This generalizes the work of Penrose in the 60's. The functional framework is adapted, so that the algebraic structures on spinors can be brought to distributions on spinors. This gives an integral formula which is simplified using the Geroch-Held-Penrose formalism. Penrose's formula on the Minkowski space-time is recovered for arbitrary spin.
On the other hand, a conformal scattering theory for a conformally invariant nonlinear wave equation is established. Using a conformal rescaling, the space-time is completed with two null hypersurfaces representing respectively the past and future endpoints of null geodesics. The asymptotic behaviour of fields is then obtained by considering the traces of solutions of the rescaled equations on these hypersurfaces. The inversibility of these trace operators is obtained by solving a characteristic Cauchy problem and the conformal scattering operator is obtained by composing these trace operators.

Mots-clefs: Relativité générale, problème de Cauchy caractéristique, équation de Dirac, formule intégrale, scattering, équation des ondes non linéaires, méthodes conformes, formalisme de Geroch-Held-Penrose.

Keywords: General relativity, problème de Cauchy caractéristique, Dirac equation, integral formula, scattering, nonlinear wave equation, conformal methods, Geroch-Held-Penrose formalism.